MATH 1150: MATHEMATICAL REASONING

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MATH 1150: Mathematical Reasoning

by Pamini Thangarajah, PhD

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Licensing

A detailed breakdown of this resource's licensing can be found in ***Back Matter/Detailed Licensing***.

Course Description:

This course explores topics in discrete mathematics including the language of logic, set theory, enumeration, probability, and statistics. Basic elements of probability and statistics will be used to solve problems involving the organization, description, and interpretation of data. Furthermore, it explores topics in Euclidean geometry.

This course is one of the required courses for the Minor in Mathematics for Elementary Education program at Mount Royal University (MRU) and was designed especially for elementary education students. The purpose of this course is to introduce future elementary (grades K-6) educators to the basic language of mathematics, and to give them an introduction to abstract concepts in set theory, geometry, probability, and statistics. The relationship of concepts to the elementary mathematics curriculum is emphasized. We started offering this course in Fall 2013. I taught this course for 5 years since its inception. I created these lecture notes to facilitate student learning. This course partially fulfills the need for mathematics to be taught as a language with reasoning and gives students spatial sense. Students will be exposed to numbers and number sense in a second course: MATH 2150 Higher Arithmetic.

The following chart gives a quick overview of further learning opportunities at MRU as they relate to this course's scope:

Chapter	If you are interested in learning more in-depth, you can choose to take this course at MRU.
3	MATH 2150 Higher Arithmetic
4	MATH 1103 Introduction to Geometry
5, 6	MATH 1102 Introduction to Statistical Reasoning

Course Learning Outcomes:

Upon successful completion of this course, students will be able to:

- show knowledge of fundamental concepts in mathematics,
- encourage mathematical investigations,
- demonstrate an understanding of mathematical reasoning,
- reflect major algebraic ideas such as algebra as a set of rules and procedures; algebra as the study of structures; algebra as the study of the relationship between quantities,
- do problem-solving by using mathematics, and
- perform mathematical calculations that involve measurements, estimation, similarity, and probability.

A Note on Formatting:

Throughout this resource, practice exercises can be found at the end of each chapter. No answer key is provided for some questions. This is to encourage students to experience mathematics as a synthetic and creative field and also to attend class to ask questions. The "Thinking Out Loud" sections are to prompt discussion - take these up with your classmates and see if you can justify your position using what you know.

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If you find any errors or mistakes, please send me an email stating the section number.

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Thumbnail: Academic knowledge is a second-order form of knowledge that seeks abstractions and generalizations based on reasoning and evidence. (Public Domain; Pearson Scott Foresman donated to the Wikimedia Foundation).

CHAPTER OVERVIEW

1: Basic Language of Mathematics

Learning Objectives

Develop the students:

- ability to understand basic logic,
- familiarity and facility with a wide range of logical statements and the connection to K-9 curriculum, and
- reasoning using truth tables and the meaning of conjectures, theorems, and counterexamples.

1.0 : Introduction to the Basic Language of Mathematics
1.1: Compound Statements
1.2: More on Logical Statements
1.3: Arguments
1.E: Basic Language of Mathematics (Exercises)

Contributor

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1.0 : Introduction to the Basic Language of Mathematics

This page is a draft and is under active development.

Introduction

Mathematical objects come into existence by definitions. These definitions must give an absolutely clear picture of the word. We don't need to prove them. We are going to state some basic facts that are needed in this course:

Basic Facts:

The collection of **counting numbers** otherwise known as the collection of **natural numbers** is usually denoted by $\mathbb{N}$. We write $\mathbb{N}=\{1,2,3,4,\ldots\}$.

The collection of the **integers** is usually denoted by $\mathbb{Z}$ and we write $\mathbb{Z}=\{\ldots,-3,-2,-1,0,1,2,3,4,\ldots\}$.

The collection of all **rational numbers** (fractions) is usually denoted by $\mathbb{Q}$ and we write $\mathbb{Q}=\left\{\frac{a}{b}: a \text{ and } b \text{ are integers}, \ b\neq 0\right\}$.

The collection of all **irrational numbers** is denoted by $\mathbb{Q}^c$.

The collection of all **real numbers** is denoted by $\mathbb{R}$. This set contains all of the rational numbers and all of the irrational numbers.

We shall assume the use of the usual addition, subtraction, multiplication, and division as operations and inequalities ($<,>,\leq,\geq$)and equality ($=$), are relations on $\mathbb{R}$.

Recall that, if a and b are real numbers, then

1. $a<b$ means that a is less than b.
2. $a>b$ means that a is greater than b.
3. $a\leq b$ means that a is less than or equal to b.
4. $a\geq b$ means that a is greater than or equal to b.

Definitions:

1. A real number is called positive if it is greater than 0.
2. A real number is called non-negative if it is greater than or equal to 0.
3. An integer n is an even number if there is an integer m such that $n=2m$.
4. An integer n is an odd number if there is an integer m such that $n=2m+1$.
5. An integer a is said to be divisible by an integer b if there is an integer m such that $a=bm$. In this case, we can say that b divides a and denoted $b|a$. Further, b is called a divisor (factor) of a.
6. A positive integer p is called prime if $p>1$ and the only positive divisors of p are 1 and p.
7. A positive integer n is called composite if there is a positive integer m such that $1<m<n$ and $m|n$.

Mathematical Statements

Thinking Out Loud:

Is $2\leq 3$ true or false? How do you know? Can you prove it?

"Knowledge is twofold and consists not only of an affirmation in what is true but in the negation of what is false." -Charles Caleb Colton, Lacon.

In any study of mathematics, language plays a vital role. Mathematical sentences are critical to any mathematical discussion, which are used to express ideas. A **mathematical statement** is a declarative sentence that is either true or false, but not both. A statement is sometimes called a **proposition**. The key to constructing a good mathematical statement is that there must be no ambiguity. To be a statement, a sentence must be true or false. It cannot be both. In mathematics, the truth of a statement is established beyond ANY doubt by a well-reasoned (logical) argument. We build upon the truths already established.

So, a sentence such as "The house is beautiful" is not a statement, since whether the sentence is true or not is a matter of opinion.

A question such as "Is it snowing?" is not a statement, because it is a question and is not declaring that something is true.

Some sentences that are mathematical in nature often are not statements because we may not know precisely what a variable represents. For example, the equation $3x+5=10$ is not a statement, since we do not know what x represents. If we substitute a specific value for x (such as $x=4$), then the resulting equation, $3x+5=10$ is a statement (which is a false statement).

"There exists a real number *x* such that $x^2+1=0$ " or "$\exists x \in \mathbb{R}\ s.t.\ x^2+1=0$ " is a statement because either such a real number exists or such a real number does not exist. In this case, this is a false statement since such a real number does not exist.

Following are some more examples:

Example 1.0.1

The following are propositions:

- Zero times any real number is zero.
- $1+1=2$.
- All birds can fly. (This is a false statement, how can you establish that?)

The following are not propositions:

- Come here.
- Who are you?
- I am lying right now. (This is a paradox, if I'm lying I'm telling the truth and if I'm telling the truth I'm lying).

Exercise 1.0.1

Which of the following are statements (propositions):

1. I like sports cars.
2. $2+3=6$.
3. Where are you?

Answer

only the second one.

Notations & Definitions

- A **mathematical statement** is a declarative sentence that is either true or false, but not both. A statement is sometimes called a **proposition**.
- $\exists$: mathematical notation for "**there exists**".
- $\in$: mathematical notation indicating a term's inclusion in a set or category.
- $\mathbb{R}$: represents the set of Real numbers.
- $s.t.$ and: represents such that.

1.1: Compound Statements

We can make a new statement from other statements; we call these **compound propositions** or **compound statements**.

Example 1.1.1:

1. It is not the case that all birds can fly. (This is the negation of the statement all birds can fly).
2. $1+1=2$ and "All birds can fly". (Here the connector "and" was used to create a new statement).

Note the following four basic ways to start with one or more propositions and use them to make a more elaborate compound statement. If p and q are statements
then here are four compound statements made from them:

1. $\neg p$, Not p (i.e. the negation of p),
2. $p \wedge q$, $p\, and\, q$,
3. $p \vee q$, $p\, or\, q$ and
4. $p \rightarrow q$, $If\, p\, then\, q$.

Example 1.1.2:

If $p =$ "You eat your supper tonight" and $q =$ "You get desert". Then

1. Not p is "You don't eat your supper tonight".
2. $p\, and\, q$ is "You eat your supper tonight and you get desert".
3. $p\, or\, q$ is "You eat your supper tonight or you get desert".
4. $If\, p\, then\, q$ is "If you eat your supper tonight then you get dessert."

In English, we know these four propositions don't say the same thing. In logic, this is also the case, but we can make that clear by displaying the truth value possibilities. It is common to use a table to capture the possibilities for truth values of compound statements. We call such a table a truth table. Below are the possibilities: the first is the least profound. It says that a statement p is either true or false.

p
T
F

Negation

Truth tables are more useful in describing the possible truth values for various compound propositions. Consider the following truth table:

p	$\neg p$
T	F
F	T

The table above describes the truth value possibilities for the statements p and $\neg p$, or "not p". As you can see, if p is true then $\neg p$ is false and if p is false, the negation (i.e. not p) is true. $\neg$ is the mathematical notation used to mean "not."

Example 1.1.3:

Consider the statement p: $1+1=3$.

Statement p can either be true or false, not both.

$\neg p$ is "not p," or the negation of statement p.

$\neg p$ is $1+1 \neq 3$.

You can see that the negation of a proposition affects only the proposition itself, not any other assumptions.

Conjunction

Conjunction statements use two or more propositions. If two or more simple propositions are involved the truth table gets bigger. Below is the truth table for "and," otherwise known as a conjunction. When is an and statement true? As the truth table indicates, only when both of the component propositions are true is the compound conjunction statement true:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example 1.1.4:

Consider statements $p := 1+1=2$ and $q := 2<5$.

Note that, $p \wedge q$ is true only if both p and q are both true.

Since statements p and q are both true, $p \wedge q$ is true.

Disjunction

Disjunction statements are compound statements made up of two or more statements and are true when one of the component propositions is true. They are called "Or Statements." In English, "or" is used in two ways:

1. If a person is looking for a house with 4 bedrooms or a short commute, a real estate agent might present houses with either 4 bedrooms or a short commute or both 4 bedrooms and a short commute. This is called an **inclusive or**.
2. If a person is asked whether they would like a Coke or a Pepsi, they are expected to choose between the two options. This is an **exclusive or**: "both" is not an acceptable case.

In logic, we use **inclusive or** statements

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

The p or q proposition is only false if both component propositions p and q are false.

Example 1.1.5:

Consider the statement $2 \leq -3$

The statement reads "2 is less than **or** equal to -3", or "$2 < -3 \vee 2 = -3$ " and can be broken into two component propositions:

1. Proposition p: $2 < -3$ (False)
2. Proposition q: $2 = -3$ (False)

Because propositions p and q are **both** false, the statement is false.

Example 1.1.6:

Consider the statement $2 \leq 5$

The statement's two component propositions are:

1. Proposition p: $2 < 5$ (True)
2. Proposition q: $2 = 5$ (False)

Since proposition p is true, the statement is true.

Conditional Statements

Consider the "if p then q" proposition. This is a **conditional statement**. Read the statements below. If these statements are made, in which instance is one lying (i.e. when is the overall statement false)?

Suppose, at suppertime, your mother makes the statement “If you eat your broccoli then you’ll get dessert.” Under what conditions could you say your mother is lying?

1. If you eat your broccoli but don't get dessert, she lied!
2. If you eat your broccoli and get dessert, she told the truth.
3. If you don’t eat your broccoli and you don’t get dessert she told you the truth.
4. If you don’t eat your broccoli but you do get dessert we still think she told the truth. After all, she only outlined one condition that was supposed to get you desert, she didn’t say that was the only way you could earn dessert. Maybe you had cauliflower instead.

Note that the order in which the cases are presented in the truth table is irrelevant. The cases themselves are important information, not their order relative to each other.

p	q	$p \to q$
T	F	F
T	T	T
F	F	T
F	T	T

It is important to notice that, if the first proposition is false, the conditional statement is true by default. A conditional statement is defined as being true unless a true hypothesis leads to a false conclusion.

Example 1.1.7:

Consider the statement "If a closed figure has four sides, then it is a square." This is a false statement - why?

We can prove it using a **counter-example**: we draw a four-sided figure that is not a square. So there!

Example 1.1.8:

Consider the statement "If $2 = 3$, then $5 = 2$"

Since $2 \neq 3$, it does not matter if $5 = 2$ is true or not, the conditional statement as a whole is true.

The converse of a conditional statement

Let P be a statement if p then q. Then the converse of P is if q then p.

Example 1.1.9:

Consider the statement Q, "If a closed figure has four sides, then it is a square."

Then the converse of Q is "If it is a square then it is a closed figure with four sides".

The contrapositive of a Conditional Statement

Let P be a statement if p then q. Then the contrapositive of P is if $\neg q$ then $\neg p$.

Example 1.1.10:

Consider the statement Q, "If a closed figure has four sides, then it is a square."

Then the converse of Q is "If it is not a square then it is not a closed figure with four sides".

Bi-Conditional Statements

Bi-conditional statements are conditional statements which depend on both component propositions. They read "p if and only if q" and are denoted $p \leftrightarrow q$ or "p iff q", which is logically equivalent to $(p \rightarrow q) \wedge (q \rightarrow p)$. These compound statements are true if both component propositions are true or both are false:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Example 1.1.11:

Consider the statement: "Two lines are perpendicular if and only if they intersect to form a right angle."

The component propositions are:

1. p: Two lines are perpendicular
2. q: [The lines] intersect to form a right angle

Logically, we can see that if two lines are perpendicular, then they must intersect to form a right angle. Also, we can see that if two lines form a right angle, then they are perpendicular.

If two lines are not perpendicular, then they cannot form a right angle. Conversely, if two lines do not form a right angle, they cannot be perpendicular. This is why, if both propositions in a biconditional statement are false, the statement itself is true!

Logically Equivalent Statements

Once we know the basic statement types and their truth tables, we can derive the truth tables of more elaborate compound statements. Below is the truth table for the proposition, **not p or (p and q)**. First, we calculate the truth values for not p, then p and q and finally, we use these two columns of truth values to figure out the truth values for not p or (p and q).

p	q	$\neg p$	$p \wedge q$	$\neg p \vee (p \wedge q)$
T	T	F	T	T
T	F	F	F	F
F	T	T	F	T
F	F	T	F	T

So the proposition "not p or (p and q)" is only false if p is true and q is false. Does this seem familiar?

"If p then q" is only false if p is true and q is false as well.

p	q	$p \rightarrow q$
T	T	T
T	F	F

p	q	$p \to q$
F	T	T
F	F	T

This has some significance in logic because if two propositions have the same truth table they are in a logical sense equal to each other – and we say that they are **logically equivalent.** So: $\neg p \vee (p \wedge q) \equiv p \to q$, or "Not p or (p and q) is equivalent to if p then q."

Example 1.1.12:

Prove or disprove: for any mathematical statements p, q and r, $p \to (q \vee r)$ is logically equivalent to $\neg r \to (p \to q)$.

p	q	r	$q \vee r$	$p \to (q \vee r)$	$\neg r$	$p \to q$	$\neg r \to (p \to q)$
T	T	T	T	T	F	T	T
T	T	F	T	T	T	T	T
T	F	T	T	T	F	F	T
T	F	F	F	F	T	F	F
F	T	T	T	T	F	T	T
F	T	F	T	T	T	T	T
F	F	T	T	T	F	T	T
F	F	F	F	T	T	T	T

Hence, $p \to (q \vee r)$ is logically equivalent to $\neg r \to (p \to q)$.

Tautologies and Contradictions

There are two cases in which compound statements can be made that result in either always true or always false. These are called **tautologies** and **contradictions**, respectively. Let's consider a tautology first, and then a contradiction:

Example 1.1.13:

Consider the statement "$(2=3) \vee (2 \neq 3)$ ":

There are two component propositions:

1. p: $2=3$
2. $\neg p$: $2 \neq 3$

Clearly, this statement is a tautology.

Let's make a truth table for general case $p \vee (\neg p)$:

p	$\neg p$	$p \vee (\neg p)$
T	F	T
F	T	T

As you can see, no matter what we do, this statement is always true. It is a **tautology**. Careful! This is not to say that this statement makes logical sense in English, but rather that, using logical mathematics, this statement is always true.

Example 1.1.14:

Consider the statement "2 is even $\wedge$ 2 is odd"

There are two component propositions:

1. p: 2 is even
2. $\neg p$: 2 is odd

Clearly, this statement is a contradiction.

Let's make a truth table for general case $p \wedge (\neg p)$:

p	$\neg p$	$p \wedge (\neg p)$
T	F	F
F	T	F

As you can see again, no matter what we do, this statement will always be false. It is a **contradiction**. These make more sense in English: 2 cannot be both even and odd, after all! Still, what matters is what we decide using logical mathematics.

Summary:

Operation	**Notation**	**Summary of truth values**
Negation	$\neg p$	The opposite truth value of p
Conjunction	$p \wedge q$	True only when both p and q are true
Disjunction	$p \vee q$	False only when both p and q are false
Conditional	$p \rightarrow q$	False only when p is true and q is false
Biconditional	$p \leftrightarrow q$	True only when both p and q are true or both are false

Notations & Definitions:

- Negation: $\neg$ or "**not**"
- Conjunction: $\wedge$ or "**and**"
- Disjunction: $\vee$ or "**or**"
- Conditional: $\rightarrow$ or "**implies**" or "**if/then**"
- Bi-Conditional: $\leftrightarrow$ or "**if and only if**" or "**iff**"
- Counter-example: An example that disproves a mathematical proposition or statement.
- Logically Equivalent: $\equiv$ Two propositions that have the same truth table result.
- Tautology: A statement that is always true, and a truth table yields only true results.
- Contradiction: A statement which is always false, and a truth table yields only false results.

1.2: More on Logical Statements

Properties

The following are some of the most frequently used logical equivalencies when writing mathematical proofs.

Theorem 1.2.1

Let p and q be statements. Then the following statements are true:

1. $p \vee p \equiv p$, and $p \wedge p \equiv p$.
2. $p \vee q \equiv q \vee p$ and $p \wedge q \equiv q \wedge p$.
3. $\neg(p \vee q) \equiv \neg p \wedge \neg q$ and $\neg(q \vee p) \equiv \neg q \wedge \neg p$.
4. $p \rightarrow q \equiv \neg q \rightarrow \neg p$.
5. $p \rightarrow q \equiv \neg p \vee q$.
6. $\neg(\neg p) \equiv p$.
7. $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$.

Predicate Logic

If we add the words “every”, “there is”, “all” and “some” to the list of logic terms we would get what is known as the predicate logical system.

The two quantifiers are:

1. For all (every) x, $P(x)$, is denoted by $\forall x P(x)$.
2. There exists (at least one) x such that $P(x)$, is denoted by $\exists x P(x)$.

Example 1.2.1:

1. Every student in this class has studied high school mathematics. By using notation, we can write, $\forall x\ P(x)$, where $P(x)$:= x has studied high school mathematics.
2. For every integer x, there exist an integer y such that $x+y=x$. By using notation, we can write, $\forall\, integer\, x, \exists\, integer\, y$ such that $x+y=x$.

Example 1.2.2:

Consider the form “X dislikes Y”.

If both variables are universally quantified, it translates as “For all X, for all Y, X dislikes Y.” In English, "Everyone dislikes everyone."

If the first variable is universally quantified and the second is existentially quantified it translates as, “For all X, there is a Y (such that) X dislikes Y”. In English, “Everyone dislikes someone".

Example 1.2.3:

For every X, there is a Y such that Y is X’s mother.

For every Y, there is an X such that Y is X’s mother.

The first statement says “Everyone has a mother”, while the second says “Everyone is a mother".

Negating statement with Quantifiers:

1. Let p be the statement $\forall x P(x)$. Then $\neg\, p$ is $\exists x, \neg\, P(x)$.
2. Let q be the statement $\exists x P(x)$. Then $\neg\, q$ is $\forall x, \neg\, P(x)$.

Example 1.2.4:

1. Every student in this class has studied high school mathematics. By using notation, we can write, $\forall x\, P(x)$, where $P(x)$:= x has studied high school mathematics.

Negation: There is a student in this class has not studied high school mathematics.

2. For every integer x, there exist an integer y such that $x+y=x$. By using notation, we can write, $\forall integer\, x, \exists integer\, y$ such that $x+y=x$.

Negation: For every integer y, there exist an integer x such that $x+y \neq x$.

Compound statements with quantifiers

Example 1.2.5:

Let Q be the statement: For all real numbers a and b, if $a+b$ is irrational or $a-b$ is irrational then a is irrational and b is irrational.

1. Write the contrapositive of Q.

Answer

For all real numbers a and b, if a is rational or b is rational then $a+b$ is rational and $a-b$ is rational.

2. Write the converse of Q.

Answer

For all real numbers a and b, if a is irrational and b is irrational then $a+b$ is irrational or $a-b$ is irrational.

3. Write the negation of Q.

Answer

There exist real numbers a and b, such that $a+b$ is irrational or $a-b$ is irrational, but a is rational or b irrational.

Terminology: Theorem, Q.E.D, and Conjecture

Theorems and Conjectures

- A **theorem** is a mathematical statement that has been proved using, and built upon, other statements, theorems, and standard axioms.
- A **conjecture** is a mathematical statement that is thought to be true but has not yet been proven formally in the field.

Thinking out loud

Can you think of a theorem and a conjecture?

1.3: Arguments

Logic is the study of the methods and principles of reasoning. An **argument** is a set of facts or assumptions, called **premises**, used to support a conclusion. For a logical argument to be **valid**, it is the case that, if the premises are true then the conclusion **must** be true.

Argument

Definition: Argument

An **argument** is a set of statements called **premises** together with a **conclusion**. An argument consisting of two premises and a conclusion is called **a syllogism**.

Example 1.3.1

1. Either a leopard is a cat or a bird. (First premise)

   It is not the case that a leopard is a bird. (Second premise)

   Therefore a leopard is a cat. (Conclusion)
2. Either a leopard is a fish or a bird. (First premise)

   It is not the case that a leopard is a bird. (Second premise)

   Therefore a leopard is a fish. (Conclusion)
3. If the Moon is a planet then the Sun is an asteroid.

The Moon is a planet.

So the Sun is an asteroid.

4. If the Earth is a planet then it is larger than a protoplanet.

   The Earth is larger than a protoplanet.

   Therefore, a leopard is a cat.
5. If the Earth is a planet then it is larger than a protoplanet.

   The Earth is larger than a protoplanet.

   Therefore, the Earth is a planet.

Arguments can be discredited if any of the premises are false (or their truth is uncertain). This is, however, not the only way an argument can be discredited. Argument 4 has true premises and a true conclusion but, nevertheless, is a poor argument. Argument 5 is an inadequate argument as well but it is harder to spot. This is because this form allows for true premises to lead to a false conclusion. Here's an example of an argument with the same form as argument 5 that we recognize is a bad argument:

Example 1.3.2

6. If the Sun is a planet then it is larger than a protoplanet.

   The Sun is larger than a protoplanet.

   Therefore, the Sun is a planet.

Valid/Invalid argument

The logical form of an argument is the second method for evaluating arguments. An argument is **valid** if and when all the premises are true. If this is the case then the conclusion must be true (i.e. if you accept the truth of the premises it forces you to accept the truth of the conclusion). We can check the validity of an argument with a truth table. When you are given a valid argument and you know the premises are true, the argument proves the conclusion to be true.

Consider the first argument arguing a leopard is a cat. It has the form:

p or q.

Not q.

Therefore p.

Is it a valid argument? Well, it's valid if whenever the premises (p or q) and (not q) are true, so the conclusion p must be true.

Here is the truth table with "p or q" and "not q" illustrated.

Table 1.3.1: Truth Table

p	q	Premise 1: $p \vee q$	Premise 2: $\neg q$	Conclusion: p
T	T	T	F	T
T	T	T	T	T
F	T	T	F	T
F	F	F	T	F

The second row of truth values is the only row where both premises are true. When both premises were true, the conclusion was true as well. This means the argument is valid. Note that we have not determined if the actual premises are correct – we've just noted that, if the premises are true it must be the case the conclusion is true as well.

The second argument is also valid because it has exactly the same form as the first (and hence the same truth table). Because its first premise is false, though, this argument is not logically sound.

Argument 4 has the form:

If p then q.

q

Therefore r.

Here is the truth table that illustrates the premises and the conclusion's possible truth values all at once. Since there are 3 simple statements (p, q and r) involved it is bigger (twice as big!) than the previous truth tables.

Table 1.3.2: Truth Table

p	**q**	**r**	**Premise 1: If p then q.**	**Premise 2: q**	**Conclusion: r**
T	T	T	T	T	T
T	T	F	T	T	F
T	F	T	F	F	T
T	F	F	F	F	F
F	T	T	T	T	T
F	T	F	T	T	F
F	F	T	T	F	T
F	F	F	T	F	F

The first, second, fifth and sixth rows of truth values all satisfy the condition that both premises are true. But row 2 and row 6 have the premises true yet the conclusion false. This is an invalid argument. Note that as soon as we see the second row of truth values we can conclude that the argument was invalid without reading through the rest of the truth table.

Of course, it gets a little tiresome to have to construct truth tables for every argument form that arises. Now that we understand the validity and are satisfied that we can determine the validity or invalidity of simple arguments, we can note classical forms of valid arguments (known as the Rules of Inference):

Table 1.3.3: Rules of Inference I

Argument type:	A. Modus ponens (affirming the hypothesis)	B. Modus tollens (denying the conclusion)	C. Disjunctive syllogism	D. Hypothetical syllogism
Premise 1	If p then q	If p then q	p or q	If p then q
Premise 2	p	Not q	Not q	If q then r
Conclusion	Therefore q	Therefore not p	Therefore p	Therefore if p then r

Table 1.3.4: Rules of Inference II

Argument type:	E. Conjunction	F. Addition	G. Simplification
Premise 1	p	p	p and q
Premise 2	q		
Conclusion	Therefore p and q	Therefore p or q (this argument has only one premise)	Therefore p

At this point we will also point out two common invalid argument types:

Table 1.3.5: common invalid argument

Argument type:	H. Affirming the Conclusion	I. Denying the Hypothesis
Premise 1	If p then q	If p then q
Premise 2	q	Not p
Conclusion	Therefore p	Therefore not q.

Both of these arguments are invalid due to the nature of conditional statements as one-directional implications.

With the valid and invalid argument forms above in mind let us consider the six arguments given, to begin with. Arguments 1 and 2 are disjunctive syllogisms and hence valid arguments. Argument 3 is modus tollens and hence is valid. Arguments 5 and 6 are examples of the affirming the conclusion argument type and hence are invalid arguments. Argument 4 is none of the above but we determined it was invalid by the truth table.

Example 1.3.3

Determine if the following arguments are valid just by looking at their form. Note that some are a combination of two or more argument types.

a. If one is a wuzzle then one is a woozle

If one is a woozle then one is a finkle

Therefore if one is a wuzzle then one is a finkle.

b. If p is a prime number larger than 2 then p is odd.

p is odd.

Therefore p is a prime number.

c. You either like Coke or you like Pepsi.

You like Coke.

So you don't like Pepsi.

d. Lizzlestipes and quadrinons.

If Lizzlestipes then fizbots.

If quadrinons then apoplexis.

Therefore fizbots and apoplexies.

e. Paul likes Banff and Jasper.

Anyone who likes Banff likes Yellowstone.

If you like Yellowstone then you like the Grand Canyon.

Therefore Paul likes the Grand Canyon.

f. If heavier objects always fall faster than lighter ones then a lead ball will fall faster than a wooden one of the same size.

It is not the case that a lead ball falls faster than a wooden one of the same size.

Therefore it is not the case that heavier objects always fall faster than lighter ones.

g. What is the origin of the Moon?

(Experts believe) Either the Moon formed elsewhere and was drawn into its present orbit by Earth's gravity OR the Moon is a companion planet to the Earth formed from the same nebula at the same time OR the Moon was once part of the Earth.

If the Moon formed elsewhere then the composition of the two objects should be dissimilar.

This is not the case.

Therefore the Moon did not form elsewhere.

If the Moon is a companion planet to the Earth then they should have similar compositions and in particular the same proportion of iron.

This is not the case.

Therefore the Moon was once part of the Earth.

Deductive argument

Note that the discussion of arguments so far has dealt with deductive arguments. In a **deductive argument**, the validity/invalidity of the argument can be determined and if we can establish with certainty that the premises are true then we know that the conclusion must be true as well. This form of reasoning, while excellent in certain circumstances, is sometimes fallible in practical applications:

1. It is often the case that there is some degree of uncertainty to the premises. For example, consider the argument for the origin of the Moon. It is a valid argument but the premises (although reasonable and widely believed among experts to be correct) can be disputed. For example, one could ask if there is a fourth explanation for the Moon's origin the experts missed. One could also point out that it is theoretically possible the Earth and Moon were formed in two completely different places but by a coincidence, the Moon happens to have the same composition as the Earth's crust.
2. It is possible to construct deductive arguments that do not apply to the general case. Consider the common childhood argument "If an animal has four legs, it is a cat." This argument is logically sound, but leads to many false conclusions.

In deductive arguments, one tends to start with general premises and arrives at a specific conclusion. Here is another example:

Example 1.3.4

All prime numbers larger than two are odd.

1098074983729749873982798649876 is an even number

It is also bigger than two.

Therefore it is not prime

Inductive argument

If one makes a case for a general conclusion from more specific premises then one is giving an **inductive argument**. Example:

Example 1.3.5

Let $x \in \mathbb{Z} \mid x > 2, x$ is prime

Since prime numbers have only themselves and 1 as factors (they are divisible only by 1 and themselves), And x is prime, $2 \nmid x$.

Since all even numbers are divisible by 2,

Thus, x is odd.

Example $1.3.6$

In study after study, it has been observed that people who smoke have a higher rate of lung cancer than those who don’t, even with sex, socioeconomic status, genetics and age taken into account.

Studies have also shown that the more one smokes, the higher one's likelihood of getting cancer.

Studies on rats have shown cigarette smoke is associated with a higher cancer rate.

Biologists experimenting with cell cultures and chemicals in cigarette smoke have discovered the basic process by which the chemicals can create cancer-causing mutations.

Therefore, smoking causes lung cancer.

Inductive arguments are a necessary part of science and life in general. Inductive arguments cannot prove (without a doubt) that their conclusion is true, but we can evaluate their strength. For example, research has made such a compelling case that smoking causes lung cancer that it is accepted as fact.

1.E: Basic Language of Mathematics (Exercises)

Exercise $1.E.1$: Statements

Which of the following are statements?

1. I am here.
2. Why am I here?
3. Life is beautiful.
4. My car is red and my house is yellow.
5. An integer is even if and only if it is divisible by 2 with no remainder.

Answer

1,5.

Thinking out loud

Is "mine" objectively true in every case? In terms of phrasing, are "my car" and "your car" different?

What about mathematically? If any given person says "my car," are they referencing the same vehicle?

Exercise $1.E.2$: Compound Statements

Let p be the statement "some people are mortal," and q be the statement "All people can reason." State, in clear English, the following cases:

1. $\neg p$
2. $\neg q$
3. $p \wedge q$
4. $\neg(p \wedge q)$

Answer

All people are not mortal, Some people can't reason, Some people are mortal and all people can reason, All people are not mortal or some people can't reason.

Exercise $1.E.3$: Truth Tables

Construct truth tables for the following statements:

1. $\neg p \wedge \neg q$

Answer

p	q	$\neg p$	$\neg p \wedge \neg q$	$\neg q$
T	T	F	**F**	F
T	F	F	**F**	T
F	T	T	**F**	F
F	F	T	**T**	T

2. $\neg q \rightarrow p$

Answer

p	q	$\neg q$	$\neg q \rightarrow p$
T	T	F	**T**

T	F	T	**T**
F	T	F	**T**
F	F	T	**F**

3. $(p \rightarrow q) \wedge (q \rightarrow p)$

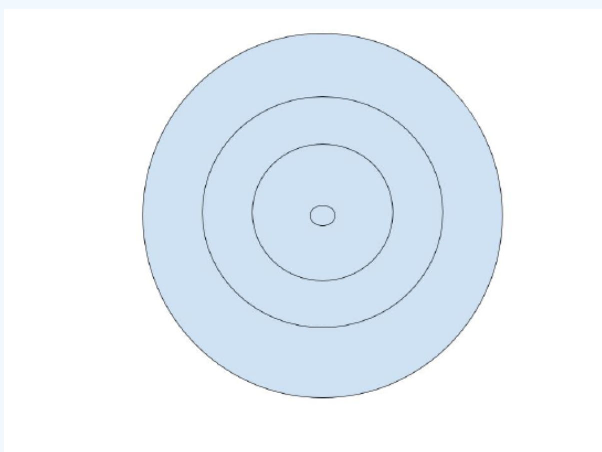
Answer

p	q	$p \rightarrow q$	$(p \rightarrow q) \wedge (q \rightarrow p)$	$q \rightarrow p$
T	T	T	**T**	T
T	F	F	**F**	T
F	T	T	**F**	F
F	F	T	**T**	T

Exercise $1.E.4$: Intuitive Reasoning

Suppose you throw four darts at a dartboard. The board has four concentric sections:

- A bull's eye, worth 10 points
- The section nearest the middle, worth 8 points
- The middle section, worth 6 points
- The outer section, worth 4 points



Supposing that all four darts thrown hit the board, what kinds of scores are possible? What kinds of scores are impossible?

Exercise $1.E.5$: Negation of Statements with Quantifiers

Find the negation of $\forall x\ \exists y\ s.t.\ x - y = 2.$

Answer

$\forall y\ \exists x\ s.t.\ x - y \neq 2.$

Exercise $1.E.6$: Logical Equivalency

Prove or disprove the following statement: The expressions $(p \vee q) \rightarrow r$ and $(p \rightarrow r) \wedge (q \rightarrow r)$ are logically equivalent.

Answer

The expressions $(p \vee q) \rightarrow r$ and $(p \rightarrow r) \wedge (q \rightarrow r)$ are logically equivalent.

Exercise $1.E.7$: True or False

Assess whether each of the following statement is true or false and justify your answer.

1. 7 is an integer and $-7 > 3$.
2. $(-5)(-2) \geq -10$.
3. If $(4)(5) = 10$ then $\frac{10}{4} = 5$.
4. If $8 < 5$ then $8 = 5$.

Answer

F, T, T, T.

Exercise $1.E.8$: Tautology

Prove or disprove: for any mathematical statements p, q and r, $(p \leftrightarrow ((\neg q) \wedge (\neg r))) \rightarrow (\neg(q \wedge r) \rightarrow p)$ is a tautology.

Exercise $1.E.9$: Logical Equivalents

Let p and q be mathematical statements. Then show that the following statements are true:

1. $p \vee q \equiv q \vee p$, and $p \wedge q \equiv q \wedge p$.
2. $\neg((p \vee q) \equiv \neg p \wedge \neg q$ and $\neg((p \wedge q) \equiv \neg p \vee \neg q$
3. $p \rightarrow q \equiv \neg q \rightarrow \neg p$
4. $p \rightarrow q \equiv \neg p \vee q$
5. $\neg(\neg p) \equiv p$
6. $p \leftrightarrow q \equiv ((p \rightarrow q) \wedge (q \rightarrow p)$

Answer

1.

p	q	$p \vee q$	$q \vee p$
T	T	**T**	**T**
T	F	**T**	**T**
F	T	**T**	**T**
F	F	**F**	**F**

Exercise $1.E.10$: Reasoning

Determine whether the following arguments are valid or invalid:

1. All polygons have angles.

A circle has no angle.

A circle is not a polygon.

2. If you don’t work hard then you won’t succeed.

You work hard.

Therefore, you will succeed.

3. If you make an A on the midterm, you won't have to take the final.

Jose did not take the final.

Therefore, Jose made an A on the midterm.

4. If a number is divisible by 8 then it is divisible by 4.

X is not divisible by 8.

Therefore x is not divisible by 4.

5. If I am rich, I would buy a cabin.

I am not rich.

Therefore I have not bought a cabin.

6. If p is a prime number larger than 2 then p is odd.

p is odd.

Therefore p is a prime number.

CHAPTER OVERVIEW

2: Basic Concepts of Sets

Learning Objectives

Course Goals and Anticipated Outcomes for This Chapter:

Develop the student:

- ability to understand the basic knowledge of set theory,
- familiarity and facility with a wide range of set-theoretical statements and the connection to K-9 curriculum, and
- reasoning using Venn diagrams and proofs.

2.0: Introduction
2.1: Subsets and Equality
2.2: Operations with Sets
2.3: Venn Diagrams and Euler Diagrams
2.5: Properties of Sets
2.E: Basic Concepts of Sets (Exercises)

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2.0: Introduction

Sets

A set is a collection of things. These things are called elements of the set. Sets are normally denoted by using capital letters, and the elements are denoted using small letters. We write $a \in A$ for "a is an element of a set A", and $a \notin A$, for "a is not an element of a set A". $\emptyset$ or $\{\}$ denotes the empty set, which contains no element.

Example 2.0.1

Let $A = \{1, 2, 3, 4, 5\}$,

Then 1 is an element of (or belongs to) set A, we write:

$1 \in A$

and 0 is not an element of A, we write:

$0 \notin A$.

Set Builder Notation

"Set builder notation" is used to express sets in which a pattern is present. Consider if set C is the set of all positive integers. Instead of writing down each one, how could we express set C in a general form?

This set would be written:

$C = \{x \in \mathbb{Z} \mid 0 < x\}$. This would read "Set C contains integers *x*, where *x* is greater than zero."

Example 2.0.2

Consider set $D = \{1, 3, 5, 7 \ldots\}$:

D consists of positive, odd integers, or $x \in \mathbb{Z},\ x > 0,\ 2 \nmid x$.

So:

$D = \{x \in \mathbb{Z} \mid x > 0,\ 2 \nmid x\}$.

Could we use any other sets to define x? Which ones would work? Which ones would not?

Example 2.0.3

Consider $\mathbb{Q}$, the set of rational numbers. How might we express $\mathbb{Q}$ in set builder notation?

Rational numbers are expressed as repeating or terminating numbers, which can be expressed as fractions: $\frac{m}{n}$.

In fractions, the denominator must not be zero: $n \neq 0$.

Also, fractions cannot have decimals as terms, so m and n must be $\in \mathbb{Z}$.

Instead of integers, if we used whole or natural numbers, we would miss out on the negative values. Thus, $m,\ n \in \mathbb{Z}$.

So:

$\mathbb{Q} = \{\frac{m}{n} \mid m,\ n \in \mathbb{Z},\ n \neq 0\}$

We can see, using set builder notation, that any number capable of being expressed as a fraction $\in \mathbb{Q}$.

Notations:

We can use set notation to specify and help describe our standard number systems. The following standard sets are given from smallest to biggest:

- $\mathbb{N}$ represents the set of all **natural numbers**: $\mathbb{N} = \{1, 2, 3, 4 \ldots\}$
- $\mathbb{W}$ represents the set of all **whole numbers**: $\mathbb{W} = \{0, 1, 2, 3 \ldots\}$

- $\mathbb{Z}$ represents the set of all **integers**: $\mathbb{Z} = \{\ldots -2, -1, 0, 1, 2 \ldots\}$ i is not used because it is used for complex numbers.
- $\mathbb{Q}$ represents the set of all **rational numbers**: $\mathbb{Q} = \{0, \pm 1, \pm \frac{1}{2}, \pm \frac{1}{3} \ldots\}$
- $\mathbb{Q}^c$ represents the set of all **irrational numbers**
- $\mathbb{R}$ represents the set of all **real numbers**
- $\mathbb{U}$ represents the **universal set**, the set to which all others are a subset.

2.1: Subsets and Equality

For this section, let A and B be sets.

Subsets

Definition: Subset

A is a subset of B, (denoted $A \subseteq B$), if every element of A is also an element of B.

TEMPLATE To prove $A \subseteq B$:

We NEED to show If $x \in A$ then $x \in B$.

Example 2.1.1:

Let A be a set. Then $\emptyset$ and A are subsets of A.

So A, $\emptyset \subseteq A$.

We can use set notation to specify and help describe our standard number systems. The following first four standard sets are given from smallest to biggest:

- $\mathbb{N}$ represents the set of all **natural numbers**: $\mathbb{N} = \{1, 2, 3, 4 \ldots\}$
- $\mathbb{W}$ represents the set of all **whole numbers**: $\mathbb{W} = \{0, 1, 2, 3 \ldots\}$
- $\mathbb{Z}$ represents the set of all **integers**: $\mathbb{Z} = \{\ldots -2, -1, 0, 1, 2 \ldots\}$ i is not used because it is used for complex numbers.
- $\mathbb{Q}$ represents the set of all **rational numbers**: $\mathbb{Q} = \{0, \pm 1, \pm\frac{1}{2}, \pm\frac{1}{3} \ldots\}$
- $\mathbb{Q}^c$ represents the set of all **irrational numbers**
- $\mathbb{R}$ represents the set of all **real numbers**
- $\mathbb{U}$ represents the **universal set**, the set to which all others are a subset.

Equal Sets

Definition: Equal Set

A is equal to B, denoted $A = B$, if $A \subseteq B$ and $B \subseteq A$.

Proper Subsets

Definition: Subset

A is a proper subset of B (denoted $A \subset B$) if $A \subseteq B$ and $A \neq B$.

Example 2.1.2:

Let $A = \{1, 3, 5\}$, $B = \{1, 5\}$, $C = \{1, 3, 5\}$, $D = \{1, 4\}$

1. $B \subset A$. since $3 \notin B$, $B \neq A$.
2. $C \subseteq A$, and $C = A$.
3. $D \nsubseteq A$ because $4 \notin A$.

Example 2.1.3:

Consider $\mathbb{N}$ and $\mathbb{W}$, the sets of natural and whole numbers.

$\mathbb{N} \subset \mathbb{W}$ because all elements of $\mathbb{N}$ are present in $\mathbb{W}$.

However, since $0 \notin \mathbb{N}$, $\mathbb{N} \nsubseteq \mathbb{W}$.

Power Sets

Definition: Power set

Let A be a set. Then the set of all subsets of A is called power set of A, and is denoted by $P(A)$.

Example 2.1.4:

Let set $A = \{Alex, Billy, Casey\}$

$P(A)$:

{Alex}	{Alex, Billy}	{Alex, Billy, Casey}
{Billy}	{Alex, Casey}	$\emptyset$
{Casey}	{Billy, Casey}	

Cardinality

Definition: Cardinality

Let A be a set. then the number of elements in the set A is called cardinality of the set A, and is denoted by $|A|$ or $n(A)$. If $n(A)$ is finite then A is called finite set, otherwise, it is called infinite set.

Example 2.1.5:

Let $A = \{1, 2, 3, 4, 5, 6, 7\}$. Then $|A| = 7$.

Example 2.1.6:

Let A be a set with $|A| = n$. Then $|P(A)| = 2^n$.

New Notations & Definitions

$\subseteq$: denotes that a set is a subset of another set.

$\subset$: denotes that a set is a proper subset of another set.

$|$: denotes for "such that" or "divides," depending on context.

{ } or $\emptyset$: denotes an empty set

Equal sets: $A = B$ if $A \subseteq B$ and $B \subseteq A$

2.2: Operations with Sets

Complement of Sets

Definition: Complement

The complement of a set is another set which contains only elements not found in the first set.

Let A be a set.

$A^c = \{x \mid x \notin A\}$

We write ^{c} to denote a complementary set.

Often, the context provides a "universe" of all possible elements pertinent to a given discussion. Suppose we have given such a set of "all" elements. Let us call it U. Then, the **complement** of a set A, denoted by A^c, is defined as $A^c = U - A$. In our work with sets, the existence of a **universal set** U is tacitly assumed.

Example 2.2.1

Consider $\mathbb{Q}$ and $\mathbb{Q}^c$, the sets of rational and irrational numbers, respectively:

$x \in \mathbb{Q} \rightarrow x \notin \mathbb{Q}^c$, since a number cannot be both rational and irrational.

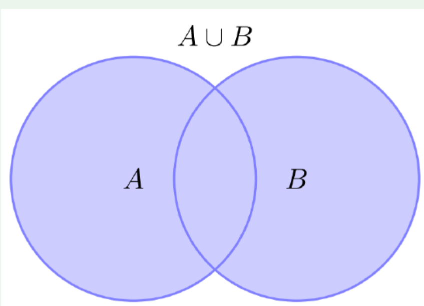
So, the sets of rational and irrational numbers are complements of each other.

Union

Definition: Union

A union of two sets creates a "united" set containing all terms from both sets.

$A \cup B = \{x \mid (x \in A) \vee (x \in B)\}$



Example 2.2.2

Let $A = \{1, 3, 5\}$ and $B = \{2, 4, 6\}$

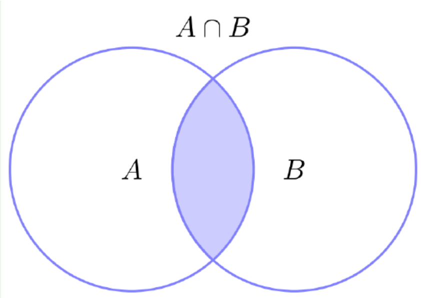
Then $A \cup B = \{1, 2, 3, 4, 5, 6\}$

Intersection

Definition: Intersection

The intersection of two sets creates a set with elements that are in both sets.

$A \cap B = \{x \mid (x \in A) \wedge (x \in B)\}$



Example 2.2.3

Let $A = \{8, 12, \frac{3}{7}, -22\}$ and $B = \{8675309, 42, 12, 8, 57\}$

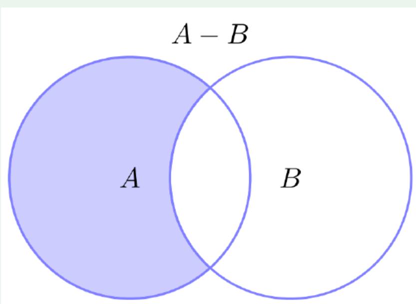
Then $A \cap B = \{8, 12\}$

Set Difference

Definition: Set difference

The difference between two sets generates a set which has no elements of the second set.

$A - B = \{x \mid (x \in A) \wedge (x \notin B)\}$



Example 2.2.4

Let $A = \{8, 12, \frac{3}{7}, -22\}$ and $B = \{8675309, 42, 12, 8, 57\}$

Then $A - B = \{\frac{3}{7}, -22\}$

The Empty Set

Definition: Empty set

The empty set is a set that has no elements. It is written $\{\}$ or $\emptyset$.

$\emptyset \subseteq A$, for any set A

The empty set has just one subset, which is itself. The empty set is also a subset of every set, since a set with no elements naturally fits into any set with elements.

Disjoints

Definition: Disjoint sets

A and B are called disjoints if $A \cap B = \emptyset$.

Example 2.2.5

Consider sets $\mathbb{Q}$ and $\mathbb{Q}^c$:

Since $\mathbb{Q} \cap \mathbb{Q}^c = \emptyset$, these sets are called disjoints.

Cartesian Product

Definition: Cartesian products

The so-called **Cartesian product** of sets is a powerful and ubiquitous method to construct new sets out of old ones.

Let A and B be sets. Then the **Cartesian product** of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), with $a \in A$ and $b \in B$. In other words,

$$A \times B = \{(a, b) \mid a \in A, b \in B\}. \tag{2.2.1}$$

An important example of this construction is the Euclidean plane $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. It is not an accident that x and y in the pair (x, y) are called the *Cartesian* coordinates of the point (x, y) in the plane.

Example 2.2.6

Let $A = \{2, 4, 6, 8\}$ and $B = \{1, 3, 5, 7\}$. then

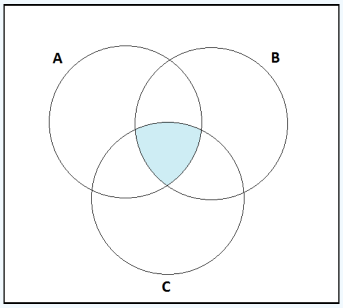
$A \times B = \{(2, 1), (4, 3), (6, 5), (8, 7)\}$

2.3: Venn Diagrams and Euler Diagrams

It is often helpful when working with sets and their operations, to use Venn diagrams:

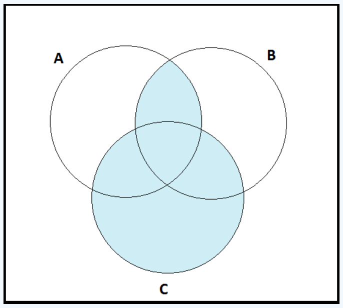
Example 2.3.1: $(A \cap B) \cap C$

Consider $(A \cap B) \cap C$:

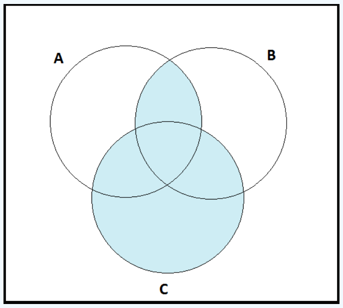


Example 2.3.2: $(A \cap B) \cup C$

Consider $(A \cap B) \cup C$:

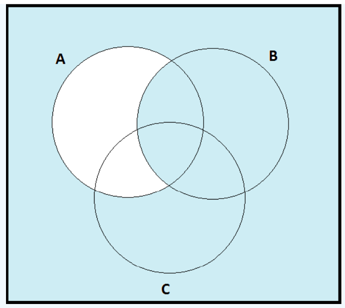


Now consider $(A \cup C) \cap (B \cup C)$:



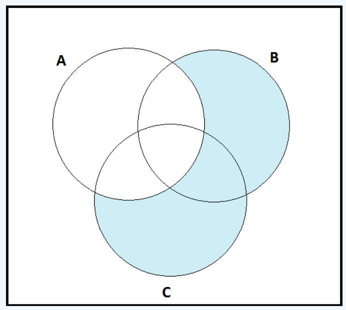
Example 2.3.3:

Consider $A^c \cup B$:



Example 2.3.4:

Consider $A^c \cap (B \cup C)$:



Another use of Venn diagrams illustrates the following result:

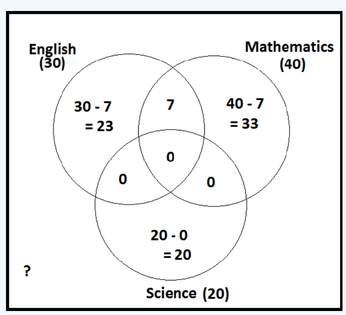
Definition: Term

Let $n(A) =$ number of elements in the set A. Then for any two finite sets A and B, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

Example 2.3.5:

A survey was taken of 150 University first-year students. 40 of them were majoring in Mathematics, 30 of them were majoring in English, 20 were majoring in Science, 7 had a double major of Mathematics and English, and none had double (triple) major with Science. How many students had majors other than Mathematics, English, or Science?

Let's use a Venn diagram to get started, shall we?



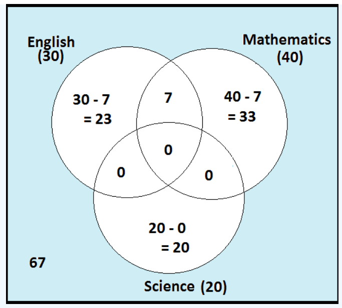
So, we know that $n = 150$, and we can calculate how many students we have currently accounted for:

$$s = \Sigma\{23 + 33 + 20 + 7\} = 83 \tag{2.3.1}$$

Using this, we can work out how many students we haven't counted yet:

$$150 - 83 = 67. \tag{2.3.2}$$

So 67 students had majors that were not in our three categories.



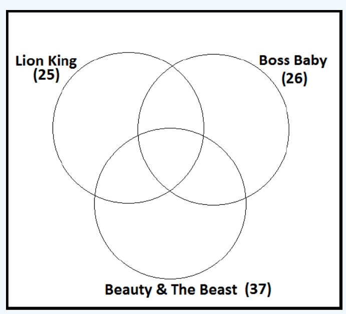
Example 2.3.6:

Suppose a group of students on a college campus is asked to compare some animated future films, and the following information is produced.

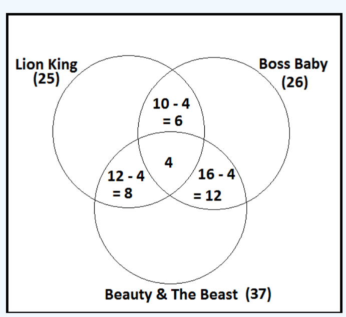
- 37 like "Beauty and the Beast"
- 26 like "The Boss Baby"
- 25 like "The Lion King"
- 16 like "Beauty and the Beast" and "The Boss Baby"
- 12 like "Beauty and the Beast" and "The Lion King"
- 10 like "The Boss Baby" and "The Lion King"
- 4 like all three films
- 5 like none of these films.

1. How many students liked "The Lion King" only?
2. How many students liked only two of the films?
3. How many students were surveyed?

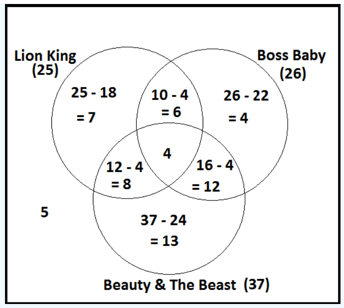
Let's use a Venn diagram to help us sort it out:



Here we know the total number of votes received for each movie, but some people may have voted twice! So, using the data, we will fill in the intersecting sections first:

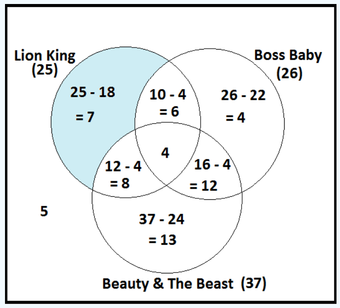


Now we can use the data to fill in the rest, before answering questions. Don't forget to include those in U that don't fall in any other category:



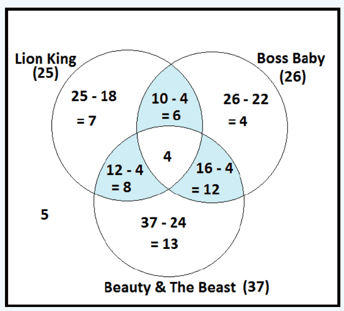
Now that we have a clear picture, we can begin to answer the questions:

1. How many students liked "The Lion King" only?



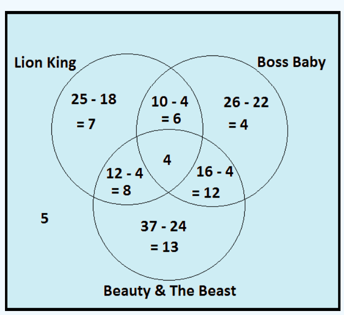
So our answer is: 'Seven students liked "The Lion King" only.'

2. How many students liked only two movies?



Our answer is: $8+6+12=26$ (the sum of the three highlighted sections), so "26 students liked only two movies."

3. What was the total number of students surveyed?



We begin by adding the values of each section:

$n=\Sigma\{7,4,13,6,12,8,4,5\}=59$

So, our answer is: "59 students were surveyed."

Summary

Let $n(A) = |A| =$ number of elements in the set A. Then for any two sets A and B, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

Euler Diagram

Euler Diagram shows relevant relationships between sets while the Venn diagram shows all possibilities.

Thinking out Loud

Can you think of an example of an Euler diagram?

2.5: Properties of Sets

Let A, B, and C be sets and U be the universal set. Then:

Commutative Law

Theorem 2.5.1: Commutative Law

For all sets A and B, $A\cup B = B\cup A$ and $A\cap B = B\cap A$

Proof

Let $x \in A\cup B$. Then $x\in A$ or $x \in B$. Which implies $x\in B$ or $x\in A$. Hence $x\in B\cup A$. Thus $A\cup B \subseteq B\cup A$. Similarly, we can show that $B\cup A \subseteq A\cup B$. Therefore, $A\cup B = B\cup A$.

Let $x\in A\cap B$. Then $x\in A$ and $x\in B$. Which implies $x\in B$ and $x\in A$. Hence $x\in B\cap A$. Thus $A\cap B\subseteq B\cap A$. Similarly, we can show that $B\cap A\subseteq A\cap B$. Therefore, $A\cap B = B\cap A$.

Distributive Law

Theorem 2.5.2: Distributive Law

For all sets A, B and C, $A\cap(B\cup C) = (A\cap B)\cup(A\cap C)$ and $A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$.

Proof

Let $x\in A\cap(B\cup C)$.

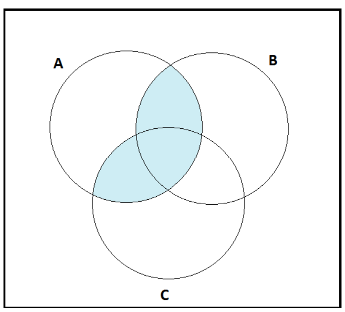
Then $x\in A$ and $x\in B\cup C$.

Thus $x\in A$ and $x\in B$ or $x\in C$.

Which implies $x\in A$ and $x\in B$ or $x\in A$ and $x\in C$.

Hence $x\in(A\cap B)\cup(A\cap C)$. Thus $A\cap(B\cup C)\subseteq(A\cap B)\cup(A\cap C)$. Similarly, we can show that $(A\cap B)\cup(A\cap C)\subseteq A\cap(B\cup C)$. Therefore, $A\cap(B\cup C) = (A\cap B)\cup(A\cap C)$.

We have illustrated using a Venn diagram:

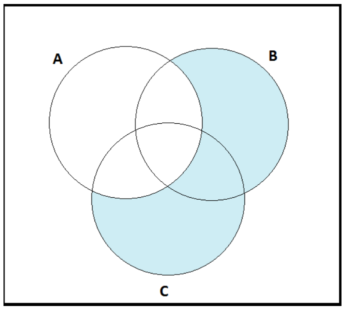


De Morgan's Laws

Theorem 2.5.3: De Morgan's Law

$(A\cup B)^c = A^c\cap B^c$ and $(A\cap B)^c = A^c\cup B^c$

We have illustrated using a Venn diagram:

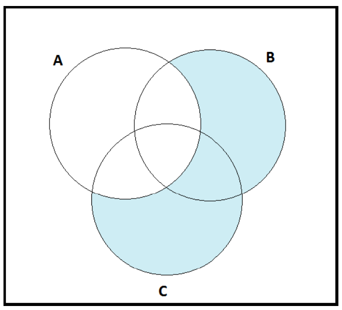


Relative Complements

Theorem 2.5.4: Relative Complements

$A\setminus(B\cup C)=(A\setminus B)\cap(A\setminus C)$ and $A\setminus(B\cap C)=(A\setminus B)\cup(A\setminus C)$.

We have illustrated using a Venn diagram:



Idempotents

Theorem 2.5.5: Idempotents

$A\cap A=A$ and $A\cup A=A$.

Identity

Theorem 2.5.6: Identity

$A\cap\emptyset=\emptyset$ and $A\cup\emptyset=A$.

Complements

Theorem 2.5.7: Complements

1. $A\cap A^c=\emptyset$ and $A\cup A^c=U$.
2. $(A^c)^c=A$.
3. $\emptyset^c=U$.
4. $U^c=\emptyset$.

2.E: Basic Concepts of Sets (Exercises)

Exercise $2.E.1$: Set Operations

Let $A=\{1,5,31,56,101\}$, $B=\{22,56,5,103,87\}$ $C=41,13,7,101,48\}$ and $D=\{1,3,5,7\ldots\}$

Give the sets resulting from:

1. $A\cap B$
2. $C\cup A$
3. $C\cap D$
4. $(A\cup B)\cup(C\cup D)$

Answer

1. $A\cap B=\{5,56\}$
2. $C\cup A=\{1,5,7,13,31,41,48,56,101\}$
3. $C\cap D=\{7,13,41,101\}$
4. $(A\cup B)\cup(C\cup D)$

Exercise $2.E.2$: True or False

1. $7\in\{6,7,8,9\}$
2. $5\notin\{6,7,8,9\}$
3. $\{2\}\not\subseteq\{1,2\}$
4. $\emptyset\not\subseteq\{\alpha,\beta,x\}$
5. $\emptyset=\{\emptyset\}$

Answer

T,T,F,F,F

Exercise $2.E.3$: Subsets

List all the subsets of:

1. $\{1,2,3\}$
2. $\{\phi,\lambda,\Delta,\mu\}$
3. $\{\emptyset\}$

Answer

1. $\{\{1,2,3\},\{1,2\},\{1,3\},\{2,3\},\{1\},\{2\},\{3\},\emptyset\}$
3. $\{\{\emptyset\},\emptyset\}$

Exercise $2.E.4$: Venn Diagram

A survey of 100 university students found the following data on their food preferences:

- 54 preferred Italian cuisine
- 29 preferred Asian-style cooking
- 16 preferred both Italian and Asian-style foods
- 19 preferred both Asian-style and Indian dishes
- 10 preferred both Italian and Indian cuisines
- 5 liked them all

- 11 did not like any of the options

How many students preferred:

1. Only Indian food?
2. Only Italian food?
3. Only one food?

Exercise $2.E.5$: Symbols

Assume that the universal set is the set of all integers.
Let
$A=\{-7,-5,-3,-1,1,3,5,7\}$
$B=\{x\in \mathbf{Z} | x^2<9\}$
$C=\{2,3,4,5,6\}$
$D=\{x\in \mathbf{Z} | x\leq 9\}$

In each of the following fill in the blank with most appropriate symbol from $\in, \notin, \subset, =, \neq, \subseteq$, so that resulting statement is true.

A-----D
3-----B
9-----D

$\{2\}$-----C^c
$\emptyset$-----D
A-----C
B-----C
C-----D
0-----$A\cap D$
0-----$A\cup D$

Exercise $2.E.6$: Prove or disprove

Given subsets A, B, C of a universal set U, prove the statements that are true and give counter examples to disprove those that are false.

1. $A-(B\cap C)=(A-B)\cup(A-C)$.
2. If $A\cap B=A\cap C$ then $B=C$.
3. If $A\cup B=A\cup C$ then $B=C$.
4. $A-(B-C)=(A-B)-C$.
5. If $A\times B\subseteq C\times D$ then $A\subseteq C$ and $B\subseteq D$.
6. If $A\subseteq C$ and $B\subseteq D$ then $A\times B\subseteq C\times D$.

Exercise $2.E.7$: Set operations

Let $A=\{r,e,a,s,o,n,i,g\}$, $B=\{m,a,t,h,e,t,i,c,l\}$ and C = the set of vowels. Calculate:

1. $A\cup B\cup C$.
2. $A\cap B$.
3. C^c.

Exercise $2.E.8$: Prove or disprove

Given subsets A, B, C of a universal set U, prove the statements that are true and give counter examples to disprove those that are false.

1. $P(A\cup B)=P(A)\cup P(B)$.
2. $P(A\cap B)=P(A)\cap P(B)$.

3. $P(A^c) = (P(A))^c$
4. $P(A - B) = P(A) - P(B)$.

Exercise 2.E.9: Equal Sets

Consider the following sets:

$A = \{x \in \mathbb{Z} | x = 2m, m \in \mathbb{Z}\}$ and $B = \{x \in \mathbb{Z} | x = 2(n-1), n \in \mathbb{Z}\}$.

Are A and B equal? Justify your answer.

Exercise 2.E.10: Product of Sets

Let $A = \{1, 3, 5\}$, and
$B = \{a, b\}$.

Then

1. Find $A \times B$ and $B \times A$.
2. Are $A \times B$ and $B \times A$ equal? Justify your answer.

CHAPTER OVERVIEW

3: Number Patterns

Course Goals and Anticipated Outcomes for This Chapter:

Develop the student:

1. ability to understand number patterns and predict the pattern,
2. familiarity and facility with a wide range of number patterns and the connection to K-9 curriculum, and
3. reasoning using induction and also using finite difference Calculus.

THINKING OUT LOUD

Consider the following sequence of numbers in which only the first two terms are given: 1,3,$\cdots$,$\cdots$. Create four different number patterns having the first two terms as 1,3, by writing out the next four terms. In each case explain the rule for your pattern. What happens if the first four terms are given as $1, 3, 5, 7, \cdots$: How many possibilities are there?

THINKING OUT LOUD

What is the perimeter of the design by joining n regular hexagons in a row? How can you prove your prediction?

Numbers can be organized into many different sequences. Most of these sequences have patterns which can be used to predict the next number in the pattern. Misunderstandings may occur when we list a few numbers in the sequence. For example, $3, 5, 7.$, the next term could be either 9 (sequence of odd integers) or 11 (sequence of prime numbers). Therefore it is wise to define sequences in terms of an explicit formula for the n^th term.

There are many types of patterns, but we will be looking at the following:

- Arithmetic sequences
- Finite sums of arithmetic sequences
- Geometric sequences
- Finite sums of geometric sequences
- other types of sequences

All sequences, regardless of how they progress, have **terms**. To denote which term we wish to consider, we use n. So, if we say that $n = 3$, we are considering the third term in a sequence. The first term in a sequence is given by a. So, if we say that $a = 23$, the first term in the given sequence is 23.

So, without further ado, let's be off!

Notations

Terms: the numbers in a sequence

- When considering a specific term: $n = x$, where x is a whole number.
- The first term in a sequence: a

Topic hierarchy

3.1: Proof by Induction
3.2: ArithmeticSequences, Geometric Sequences : Visual Reasoning, and Proof by Induction
3.3: Recognising Sequences
3.4: Finite Difference Calculus
3.E: Number Patterns (Exercises)

Thumbnail: Derivation of triangular numbers from a left-justified Pascal's triangle. 9Cc BY-SA 4.0; Cmglee).

Thanks to Thomas Thangarajah for sharing his hexagonal drawing.

Contributor

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3.1: Proof by Induction

Inductive reasoning is the process of drawing conclusions after examining particular observations. This reasoning is very useful when studying number patterns. In many situations, inductive reasoning strongly suggests that the statement is valid, however, we have no way to present whether the statement is true or false, for example, Goldbach conjecture. But, in this class, we will deal with problems that are more accessible and we can often apply mathematical induction to prove our guess based on particular observations. For example, when we predict a n^{th} term for a given sequence of numbers, mathematics induction is useful to prove the statement, as it involves positive integers.

Process of Proof by Induction

There are two types of induction: regular and strong. The steps start the same but vary at the end. Here are the steps. In mathematics, we start with a statement of our assumptions and intent:

Let $p(n), \forall n \geq n_0,\ n,\ n_0 \in \mathbb{Z}_+$ be a statement. We would show that p(n) is true for all possible values of n.

1. Show that p(n) is true for the smallest possible value of n: In our case $p(n_0)$. **AND**
2. **For Regular Induction:** Assume that the statement is true for $n=k,$ for some integer $k \geq n_0$. Show that the statement is true for n = k + 1.

OR

For Strong Induction: Assume that the statement p(r) is true for all integers r, where $n_0 \leq r \leq k$ for some $k \geq n_0$. Show that p(k+1) is true.

If these steps are completed and the statement holds, we are saying that, by mathematical induction, we can conclude that the statement is true for all values of $n \geq n_0$.

We shall use the following template for proof by induction:

Template for proof by induction

In order to prove a mathematical statement involving integers, we may use the following template:

Suppose $p(n), \forall n \geq n_0,\ n,\ n_0 \in \mathbb{Z}_+$ be a statement.

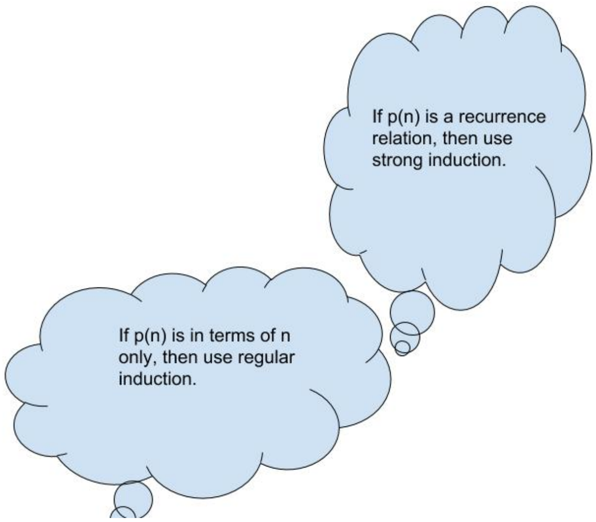
For regular Induction:

- **Base Case:** We need to show that p(n) is true for the smallest possible value of n: In our case show that $p(n_0)$ is true.
- **Induction Hypothesis**: Assume that the statement $p(n)$ is true for any positive integer $n=k,$ for s $k \geq n_0$.
- **Inductive Step:** Show that the statement $p(n)$ is true for $n=k+1.$.

For strong Induction:

- **Base Case:** Show that p(n) is true for the smallest possible value of n: In our case $p(n_0)$.
- **Induction Hypothesis**: Assume that the statement $p(n)$ is true for all integers r, where $n_0 \leq r \leq k$ for some $k \geq n_0$.
- **Inductive Step:** Show that the statement $p(n)$ is true for $n=k+1.$.

If these steps are completed and the statement holds, by mathematical induction, we can conclude that the statement is true for all values of $n \geq n_0$.



Example 3.1.1

Prove $2^n > n+4$ for $n \geq 3, n \in \mathbb{N}$.

Solution

Let $n=3$. Then $2^3 > 3+4$ is true since clearly $8>7$. Thus the statement is true for $n=3$.

Assume that $2^n > n+4$ is true for some $n=k$.

We will show that $2^{k+1} > (k+1)+4$.

Consider $2^{k+1} = 2 \cdot 2^k > 2 \cdot (k+4) = 2k+8$.

Since $2k > k+1$ and $8>4$, we have $2k+8 > (k+1)+4$.

Thus the statement is true for all $n=k$.

By induction, $2^n > n+4$ for all $n \geq 3, n \in \mathbb{Z}$.□

Exercise 3.1.1

Prove that $n < 2^n$ for $n \in \mathbb{N}$.

Answer

Hint: $k+1 < 2^k(1+1)$.

Example 3.1.2

Prove that $1+2+...+n = \dfrac{n(n+1)}{2}, \forall n \in \mathbb{Z}$.

Solution:

Base step: Choose $n=1$. Then L.H.S =1. and R.H.S $= \frac{(1)(1+1)}{2} = 1$

Induction Assumption: Assume that $1+2+...+k = \dfrac{k(k+1)}{2}$, for $k \in \mathbb{Z}$.

We shall show that $1+2+...+k+(k+1) = \dfrac{(k+1)[(k+1)+1]}{2} = \dfrac{(k+1)(k+2)}{2}$

Consider $1+2+...+k+(k+1)$

$$= \frac{k(k+1)}{2} + (k+1)$$

$$= (k+1)\left(\frac{k}{2} + \frac{1}{1}\right)$$

$$= (k+1)\left(\frac{k+2}{2}\right)$$

$$= \frac{(k+1)(k+2)}{2}.$$

Thus, by induction we have $1+2+\ldots+n = \frac{n(n+1)}{2}, \forall n \in \mathbb{Z}$.

We will explore examples that are related to number patterns in the next section.

3.2: ArithmeticSequences, Geometric Sequences : Visual Reasoning, and Proof by Induction

Arithmetic Sequences

Definition

Arithmetic sequences are patterns of numbers that increase (or decrease) by a set amount each time when you advance to a new term. You can determine the next term by adding the difference between any two terms to the final one to generate the next term. Let a be the initial term and d be the difference, then the n^{th} term of the arithmetic sequence can be expressed as $t_n = a + (n-1)d$.

Example 3.2.1:

n	n^{th} Term $= t_n$	$t_n - t_{(n-1)}$
5	11	2
6	13	2
7	15	2
8	17	2

As you can see, this sequence's terms increase by 2 each time.

Example 3.2.2:

Given the following sequence, can we determine a and d and give the sequence's general form?

1, 4, 7, 10, 13, 16, 19...

So:

- $a = 1$, because that is the first term in the sequence.
- $d = 3$, because the terms increase by 3 each time.

So the general form for the sequence is:

$t_n = 1 + (n-1)3 = 3n + 2$

Thinking Out Loud:

Is a sequence still an arithmetic sequence if the difference changes with each iteration, even if it is still added? Why or why not? What truly defines an arithmetic sequence?

Finite Sum of Arithmetic Sequences

There are two, equivalent, formulas for determining the finite sum of an arithmetic sequence. Here, we shall derive both the formulas and show how they are equal.

Example 3.2.3:

Consider $S_n = a \ + (a+d) \ + \cdots + (a + (n-2)d) + (a + (n-1)d)$

Now, $S_n = (a + (n-1)d) + (a + (n-2)d + \cdots + (a+d) \ + a$.

Then, $2S_n = n(a + a + (n-1)d) \ = n(2a + (n-1)d)$.

Hence, $S_n = \frac{n}{2}(t_1 + t_n) \ = \frac{n}{2}(2a + (n-1)d)$.

Let's illustrate these formulas by using the sequence $t_n = 5 + (n-1)2 = 2n + 3$.

Example 3.2.4:

The first formula we can use looks like this:

$$S_n = n\left(\frac{a_1 + a_n}{2}\right)$$

As we can see, this formula takes the average between the first and last terms, and multiplies by the number of terms in the series. So, if we use our series $t_n = 5 + (n-1)2$ and we want the sum for the first 15 terms, our calculation will look like this:

$$S - 15 = 15\left(\frac{5+33}{2}\right)$$

$$= 15\left(\frac{38}{2}\right)$$

$$= 15(19)$$

$$= 285$$

Example 3.2.5:

The second formula we can use looks like:

$$S_n = \frac{n}{2}(2a + (n-1)d)$$

As we can see, this method doesn't need us to know the value of the n^{th} term, just which term it is. Using our series $t_n = 5 + (n-1)2$, our calculations look like this when we are looking for the sum of the first 15 terms:

$$S = \frac{15}{2}(2(5) + ((15) - 1)(2))$$

$$= \frac{15}{2}(10 + (14)(2))$$

$$= \frac{15}{2}(38)$$

$$= 285$$

So these two methods look to be equivalent so far. Let's show that this is true in the general case:

Example 3.2.6:

Consider $S = \frac{n}{2}(2a + (n-1)d)$, where n is the number of the term that is the endpoint, a is the series' starting value, and d is the difference between any two consecutive terms.

Consider $S = n\left(\frac{a_1 + a_n}{2}\right)$

Since $a_n = t_n = a + (n-1)d$,

And $a_1 = a$,

Then $S = n\left(\frac{a + [a + (n-1)d]}{2}\right)$

$$= n\left(\frac{2a + (n-1)d}{2}\right)$$

$$= \frac{n}{2}(2a + (n-1)d)$$

Let's explore summation notation which will be useful to represent finite sums:

Sigma (Summation) Notation

A finite sum requires adding up long strings of numbers. To make it easier to write down these lengthy sums, we look at some new notation here, called **sigma notation** (also known as **summation notation**). The Greek capital letter Σ, sigma, is used to express long sums of values in a compact form. For example, if we want to add all the integers from 1 to 20 without sigma notation, we have to write

$$1+2+3+4+5+6+7+8+9+10+11+12+13+14+15+16+17+18+19+20. \tag{3.2.1}$$

We could probably skip writing a couple of terms and write

$$1+2+3+4+\cdots+19+20, \tag{3.2.2}$$

which is better, but still cumbersome. With sigma notation, we write this sum as

$$\sum_{i=1}^{20} i \tag{3.2.3}$$

which is much more compact. Typically, sigma notation is presented in the form

$$\sum_{i=1}^{n} a_i \tag{3.2.4}$$

where a_i describes the terms to be added, and i is called the $index$. Each term is evaluated, then we sum all the values, beginning with the value when $i=1$ and ending with the value when $i=n$. For example, an expression like $\sum_{i=2}^{7} s_i$ is interpreted as $s_2+s_3+s_4+s_5+s_6+s_7$. Note that the index is used only to keep track of the terms to be added; it does not factor into the calculation of the sum itself. The index is therefore called a **dummy variable**. We can use any letter we like for the index. Typically, mathematicians use *i*, *j*, *k*, *m*, and *n* for indices.

Let's try a couple of examples of using sigma notation.

Example 3.2.7: Using Sigma Notation

a. Write in sigma notation and evaluate the sum of terms 3^i for $i=1,2,3,4,5$.
b. Write the sum in sigma notation:

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}. \tag{3.2.5}$$

Solution

a. Write

$$\sum_{i=1}^{5} 3^i = 3+3^2+3^3+3^4+3^5 = 363. \tag{3.2.6}$$

b. The denominator of each term is a perfect square. Using sigma notation, this sum can be written as $\sum_{i=1}^{5} \frac{1}{i^2}$.

Exercise 3.2.1

Write in sigma notation and evaluate the sum of terms 2^i for $i=3,4,5,6$.

Hint

Use the solving steps in Example as a guide.

Answer

$\sum_{i=3}^{6} 2^i = 2^3+2^4+2^5+2^6 = 120$

The properties associated with the summation process are given in the following rule.

Rule: Properties of Sigma Notation

Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ represent two sequences of terms and let c be a constant. The following properties hold for all positive integers n and for integers m, with $1 \leq m \leq n$.

1. $\sum_{i=1}^{n} c = nc$
2. $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$
3. $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$
4. $\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$
5. $\sum_{i=1}^{n} a_i = \sum_{i=1}^{m} a_i + \sum_{i=m+1}^{n} a_i$

Proof:

We prove properties 2. and 3. here, and leave proof of the other properties to the Exercises.

2. We have

$$\sum_{i=1}^{n} ca_i = ca_1 + ca_2 + ca_3 + \cdots + ca_n = c(a_1 + a_2 + a_3 + \cdots + a_n) = c\sum_{i=1}^{n} a_i \quad .$$

3. We have

$$\sum_{i=1}^{n}(a_i + b_i) = (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) + \cdots + (a_n + b_n) \tag{3.2.7}$$

$$= (a_1 + a_2 + a_3 + \cdots + a_n) + (b_1 + b_2 + b_3 + \cdots + b_n) \tag{3.2.8}$$

$$= \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i. \tag{3.2.9}$$

□

A few more formulas for frequently found functions simplify the summation process further. These are shown in the next rule, for **sums and powers of integers**, and we will explore further in later examples.

Rule: Sums and Powers of Integers

1. The sum of n integers is given by

$$\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}. \tag{3.2.10}$$

2. The sum of consecutive integers squared is given by

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}. \tag{3.2.11}$$

3. The sum of consecutive integers cubed is given by

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}. \tag{3.2.12}$$

Proof

We leave proof (by induction) of the rules to the Exercises.

Geometric Sequences

Definition:

Geometric sequences are patterns of numbers that increase (or decrease) by a set **ratio** with each iteration. You can determine the ratio by dividing a term by the preceding one. Let a be the initial term and r be the ratio, then the n^{th} term of a geometric sequence can be expressed as $t_n = ar^{(n-1)}$.

Example 3.2.8

n	t_n	t_n/t_{n-1}
1	3	
2	6	2
3	12	2
4	24	2

So we can see that $r = 2$, since the ratio between any two consecutive terms is 2.

Example 3.2.9

Given the sequence $-3, 6, -12, 24, -48..$, can we:

1. Determine a and r
2. Express the general form of the sequence

So:

- $a = -3$, because that is the sequence's initial term.
- $r = -2$, because if we divide any term by the preceding one, that is the result.

So the general form for the sequence is:

$t_n = -3(-2)^{(n-1)}$.

Finite Sum of Geometric Sequences

Let's use the Gauss method for finding a general case for the sum of a geometric sequence:

Example 3.2.10

Let $r \neq 1$.

Consider $S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{(n-1)}$

Now, $(1)S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{(n-1)}$

$-(r)S_n = ar + ar^2 + ar^3 + \ldots + ar^{(n-1)} + ar^n$

$(1-r)S_n = a - ar^n$

$S_n = \dfrac{a(1-r^n)}{1-r}$

That is,

$$\sum_{k=0}^{(n-1)} ar^k = \frac{a(1-r^n)}{1-r}, r \neq 1. \tag{3.2.13}$$

Sum of Integers

Observe:

$1=1$

$1+2=3$

$1+2+3=6$

$1+2+3+4=10$

$1+2+3+4+5=15$

$1+2+3+\ldots+n=?$ This is the finite sum of first n positive integers. Below we have shown two ways of finding this sum:

Example 3.2.11

Let's figure out a general case for a sum of integers beginning with 1 and ending with n using Gauss' method:

Let $S_n=1+\quad 2\quad+\quad 3\quad+\ldots+n$

$+S_n=n+(n-1)+(n-2)+\ldots+1$

So $2S_n=n(n+1)$.

This is because, when you add S to itself (with the order reversed), you get $n+1$ repeated n times.

Then, $S_n=\dfrac{n(n+1)}{2}$

That is $\sum_{k=1}^{n} k=\dfrac{n(n+1)}{2}$.

Example 3.2.12

Here's that same concept being proven inductively:

Prove that $1+2+\ldots+n=\dfrac{n(n+1)}{2}$, $\forall n\in\mathbb{Z}$

Base step: Choose $n=1$. Then L.H.S =1. and R.H.S $=\frac{(1)(1+1)}{2}=1$

Induction Assumption: Assume that $1+2+\ldots+k=\dfrac{k(k+1)}{2}$, for $k\in\mathbb{Z}$.

We shall show that $1+2+\ldots+k+(k+1)=\dfrac{(k+1)[(k+1)+1]}{2}=\dfrac{(k+1)(k+2)}{2}$

Consider $1+2+\ldots+k+(k+1)$

$=\dfrac{k(k+1)}{2}+(k+1)$

$=(k+1)\left(\dfrac{k}{2}+\dfrac{1}{1}\right)$

$=(k+1)\left(\dfrac{k+2}{2}\right)$

$=\dfrac{(k+1)(k+2)}{2}$.

Thus, by induction we have $1+2+\ldots+n=\dfrac{n(n+1)}{2}$, $\forall n\in\mathbb{Z}$.

Thinking Out Loud:

Can we determine S of a sequence of integers that do not start at 1?

Sum of Positive Odd Integers

Thinking Out Loud:

The edges of an equilateral triangle are divided into n equal segments by inserting $n-1$ points. Lines are drawn through each of these points parallel to each of the three edges, forming a set of small triangles. How many of the small triangles are there? Justify your answer.

Observe:

$1=1$

$1+3=4$

$1+3+5=9$

$1+3+5+7=16$

$1+3+5+7+9=25$

$1+3+5+\ldots+(2n-1)=?,$

Example 3.2.13

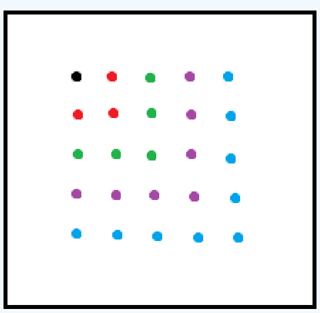
Let's look at a table of the sums of the first n postive odd integers:

n	S_n
1	1
2	4
3	9
4	16
5	25

As we can see, the sum of the first n positive odd integers $=n^2$

Example 3.2.14

Let's try a visual proof for this one as well. Remember, square numbers can be arranged into perfectly square arrays.



As we can see, when we arrange odd integers into an array (each new term is represented by a new color), we always have an array with n^2 points.

Exercise3.2.2

By using induction, prove that $1+3+5+\ldots+(2n-1)=n^2$, for all $n\geq 1$.

Sum of Positive Even Integers

Observe:

$2 = 2$

$2 + 4 = 6$

$2 + 4 + 6 = 12$

$2 + 4 + 6 + 8 = 20$

$2 + 4 + 6 + 8 + 10 = 30$

$2 + 4 + 6 + ... + 2n = ?, n \in \mathbf{N}$

Example 3.2.15

Let's try deriving this using what we already know:

If we write the sum of positive even integers as

$2 + 4 + 6 + 8 + ... + 2n$,

We see we can factor out the 2:

$2(1 + 2 + 3 + ... + n)$. This is great news! We already know the sum of a finite set of positive integers. It is $\frac{n(n+1)}{2}$

So then the sum of a series of positive even integers is:

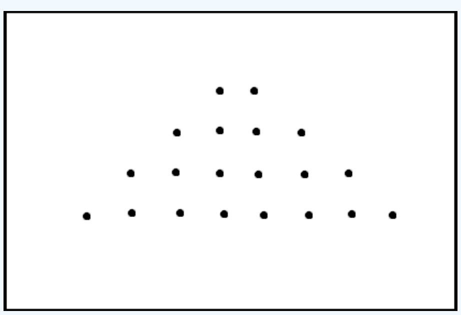
$$2\left(\frac{n(n+1)}{2}\right)$$

Or $n(n+1)$.

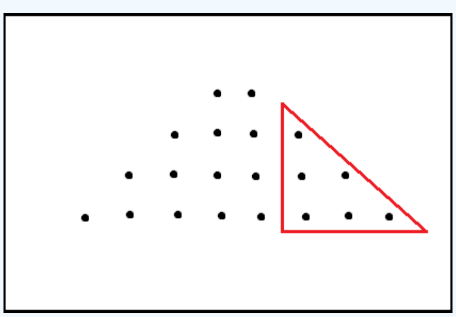
Example 3.2.16

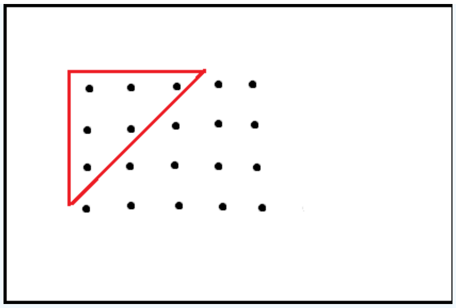
Let's try a visual proof:

Here is the sum of the first 4 positive even integers, or $n = 4$:



Now, if we move some of the points to make a rectangular array...





...we can see that, for n terms, our array is described by:

$n(n+1)$

Contributors

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- Pamini Thangarajah (Mount Royal University, Calgary, Alberta, Canada)

3.3: Recognising Sequences

For some of the sequences, we can predict the n^{th} term easily, and the explicit (general) formula can be checked using induction. In this section, we will explore this kind of sequence.

Example 3.3.1: Harmonic sequence

Find a formula for the n^{th} term of the a sequence that has the following initial terms: $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \cdots$.

Answer

$$\frac{1}{n+1}$$

Caution

Two sequences may start with the same initial terms but diverge later on.

Example 3.3.2: One's

$1 \times 11 = 11$

$11 \times 11 = 121$

$111 \times 111 = 12321$

$1111 \times 1111 = 1234321$

Can you predict the pattern?

Example 3.3.3: Perfect Squares

Find a formula for the n^{th} term of the following sequence $1, 4, 9, 16, 25, \cdots$.

Answer

n^2.

Here is another way to find the n^{th} term:

Note that the difference in the sequence $1, 4, 9, 16, 25, \cdots$ is $3, 5, 7, 9, \cdots$ and the difference in this sequence is 2. Therefore the n^{th} can be written as a quadratic formulae $pn^2 + qn + c$. p, q, c can be found using the following information:

	t_n	$d_n = t_n - t_{(n-1)}$	$d = d_n - d_{(n-1)}$
n=1	p+q+c		
n=2	4p+2q+c	3p+q	
n=3	9p+3q+c	5p+q	2p
n=4	16p+4q+c	7p+q	2p

Hence, in our case, $2p = 2$. Therefore, $p = 1$. Now $3p + q = 3$ and $p = 2$. Thus $q = 0$. Now $p + q + c = 1$, which implies $c = 0$. Hence, $t_n = n^2$.

Quadratic Sequences:

A sequence is called quadratic sequence if the differences of consecutive terms, $d_n = t_n - t_{(n-1)}$ differ by the same amount $d = d_n - d_{(n-1)}, \forall n \in \mathbf{N}$. In this case, the n^{th} term of the sequence is given by

$t_n = a + (n-1)d_1 + (n-1)(n-2)\frac{d}{2}$,

where d_1 is the first difference between first term and the second term of the sequence, and d is the common second difference. This result can be shown by using induction. We will explore this in the next section.

Below is another way to solve:

Example 3.3.4: Quadratic Sequences

Find a formula for the n^{th} term of the following sequence: $2, 6, 12, 20, \cdots$

2 6 12 20

4 6 8

2 2

Hence the n^{th} term has a term $(2/2)n^2$.

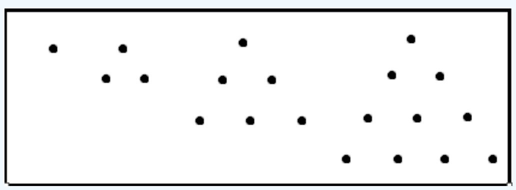
t_n	2	6	12	20
n^2	1	4	9	16
$n^2 - t_n$	1	2	3	4

Notice that $n^2 - t_n$ is an arithmetic sequence with the first term 1 and the common difference is 1. Thus $n^2 - t_n = 1 + 1(n-1)$. Hence $t_n = n^2 - n$.

Triangular numbers

Thinking Out Loud:

$1, 3, 6, 10, 15, 21, 28, . .$ are triangular numbers: you can arrange them in a triangular array:



$T_n = n^{\text{th}}$ triangular number. What is the hundredth triangular number?

Example 3.3.5: Hexagonal Tilling (Centered hexagonal numbers)

Find the n^{th} term of the sequence $1, 7, 19, 37, \cdots$:

By Incnis Mrsi (Own work) [CC0], via Wikimedia Commons

There are 6 triangular numbers and 1 center, Therefore \(6 \frac{(n -1)(n)}{2} + 1 = 3(n -1)(n) + 1 = 3(n^2 -n) + 1 = 3n^2 -3n + 1 .\)

Example 3.3.6:

Consider the sequence of tilling using hexagons. The hexagon numbers are the sequence 1, 6, 15, $\cdots$. Predict the nth term. Explain your prediction.

A.	Term # (n)	1	2	3	4
B.	# of hexagons	1	6	15	28

C.	B/A	1	3	5	7
D.	2(n)-1	1	3	5	7

C = 2n-1

Since we know that (A)C = B, we can conclude that the term number (A)=n, multiplied by C (2n-1) will give you the corresponding number in the sequence (B).

Therefore:

$$t_n = n(2n-1) = 2n^2 - n. \tag{3.3.1}$$

Tower of Hanoi

According to the legend of the Tower of Hanoi (formerly the "Tower of Brahma" in a temple in the Indian city of Benares), the temple priests are to transfer a tower consisting of 64 fragile disks of gold from one part of the temple to another, one disk at a time. The disks are arranged in order, no two of them the same size, with the largest on the bottom and the smallest on top. Because of their fragility, a larger disk may never be placed on a smaller one, and there is only one intermediate location where disks can be temporarily placed. It is said that before the priests complete their task the temple will crumble into dust and the world will vanish in a clap of thunder.

Let A, B and C be the posts. Then •1 disk: 1 move

Move 1: move disk 1 to post C

•2 disks: 3 moves

Move 1: move disk 2 to post B
Move 2: move disk 1 to post C
Move 3: move disk 2 to post C

•3 disks: 7 moves

Move 1: move disk 3 to post C
Move 2: move disk 2 to post B
Move 3: move disk 3 to post B
Move 4: move disk 1 to post C
Move 5: move disk 3 to post A
Move 6: move disk 2 to post C
Move 7: move disk 3 to post C

The number of moves needed to transfer n disks from post-A to post C is $2M+1$, where M is the number of moves needed to transfer $n-1$ disks from post A to post C . This is called a recursive sequence. Can we able to guess a formula depending on n only?

Number of Disks	Min. number of Moves
1	1
2	3
3	7
4	15
5	31

From this pattern, we can guess the formula for finding the minimum number of moves it takes to transfer n disks from post-A to post C is: $2^n - 1$. This guess can be proved by using induction.

Recursive Sequences

Definition

A recurrence relation for a sequence $\{a_n\}$ is formula that relates to each term a_n to its predecessors $a_0, a_1, \cdots, a_{n-1}$.

Thinking Out Loud:

A tiling consists of covering a region using tile pieces from some given set so that the region is completely covered without overlaps. How many ways can you arrange the 2 x 1 dominoes to cover the 2 x n checkerboard?

Fibonacci Sequences

By Ein_Hase_mit_blauem_Ei.svg: MichaelFrey & Sundance Raphael derivative work: HB (Ein_Hase_mit_blauem_Ei.svg) [CC BY-SA 3.0 (https://creativecommons.org/licenses/by-sa/3.0) or GFDL (http://www.gnu.org/copyleft/fdl.html)], via Wikimedia Commons

Fibonacci sequence: $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, , \ldots$

Let $F_n = n^{th}$ Fibonacci Number.The n term in the Fibonacci sequence is obtained by adding the previous two terms.

That is, $F_n = F_{(n-1)} + F_{(n-2)}, n > 2,\ F_1 = 1,$ and $F_2 = 1$

Some facts about the Fibonacci sequence :

- The sum of the first n even-numbered Fibonacci numbers is one less than the next Fibonacci number.
- The only square Fibonacci numbers are 0, 1 and 144.
- The sum of the first n odd-numbered Fibonacci numbers is the next Fibonacci number.

Example 3.3.9:

Consider the Fibonacci sequence: $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, , \ldots$

The squares of the Fibonacci sequence: $1, 1, 4, 9, 25, 64, 169 \ldots$...Consider the sum of the squares of consecutive terms, $F^2_{(n-1)} + F^2_{(n-2)}$, for $n > 2$. That is:

1+1=2

1+4 =5

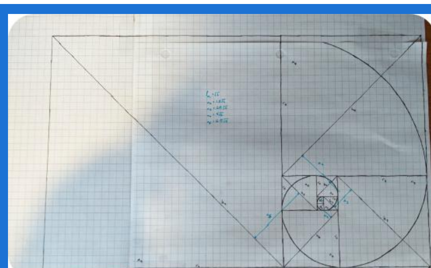
4+9=13

What can you say about the resulting number?

Answer

It is Fibonacci. In fact, $F_{(2n-3)} = F^2_{(n-1)} + F^2_{(n-2)}$.

Example 3.3.11: More Fibonacci



Let a_n be the Fibonacci sequence. That is $a_n = a_{n-1} + a_{n-2}, n \in \mathbb{N},$ with $a_1 = 1, a_0 = 0$, where $\mathbb{N}$ be the set of all natural numbers.
Consider the Fibonacci squares illustrated in the figure:

Let h_n be the shortest (perpendicular) distance between n^{th} parallel diagonal vectors. Consider the figure:

Comparing the the area of trapezoids, we get

$$h_n = \frac{a_n^2 - a_{n-1}^2}{\sqrt{2}(a_n - a_{n-1})} = \frac{1}{\sqrt{2}}(a_n + a_{n-1}).$$

Hence, h_n is Fibonacci like sequence.

- Hexagonal numbers by Incnis Mrsi (Own work) [CC0], via Wikimedia Commons
- Tower of Hanoi by André Karwath aka Aka (Own work) [CC BY-SA 2.5 (https://creativecommons.org/licenses/by-sa/2.5)], via Wikimedia Commons
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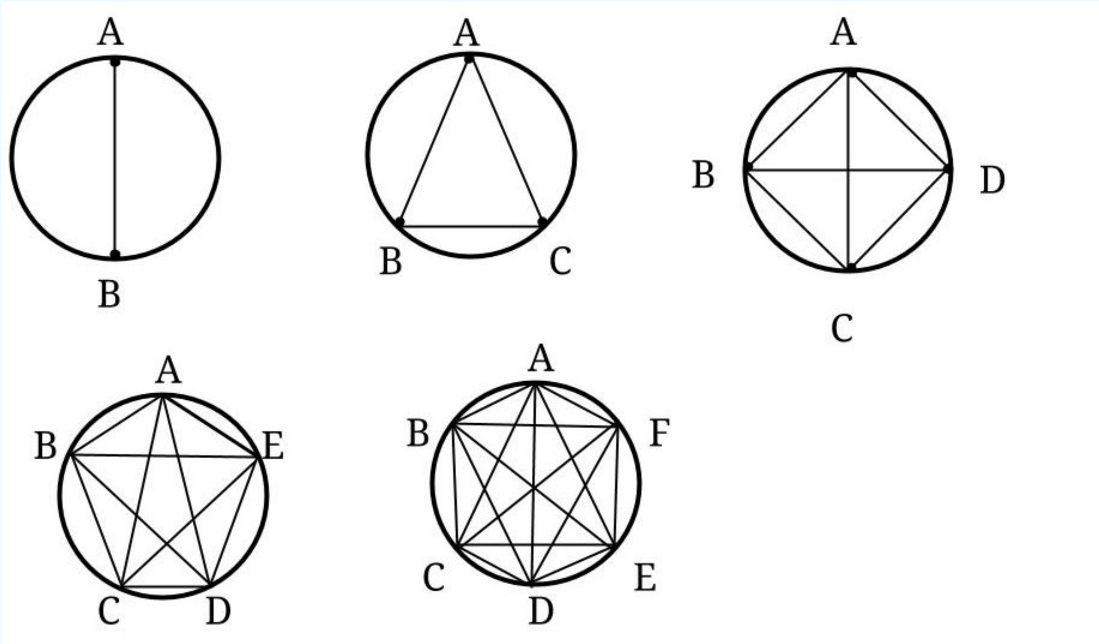
3.4: Finite Difference Calculus

This page is a draft and is under active development.

In this section, we will explore further to the method that we explained at the introduction of Quadratic sequences.

Example 3.4.1

Create a sequence of numbers by finding a relationship between the number of points on the circumference of a circle and the number of regions created by joining the points.



Number of points on the circle	0	1	2	3	4	5	6	7	8
Number of regions	1	1	2	4	8	16	31	57	99

Difference operator

Notation:

Let $a_n, n=0,1,2,\cdots$ be a sequence of numbers. Then the first difference is defined by $\Delta a_n = a_{n+1} - a_n, n=0,1,2,\cdots$.

The second difference is defined by $\Delta(\Delta a_n) = \Delta^2 a_n = \Delta a_{n+1} - \Delta a_n, n=0,1,2,\cdots$.

Further, k^{th} difference is denoted by $\Delta^k a_n, n=k,\cdots$. We shall denote that $\Delta^0 a_n = a_n$.

Example 3.4.2

Let $a_n = n^2$, for all $n \in \mathbb{N}$. Show that $\Delta^2 a_n = 2$.

Solution

$\Delta a_n = (n+1)^2 - n^2 = 2n+1,$

$\Delta^2 a_n = (2(n+1)+1) - (2n+1) = 2.$

Rule

Let $k \in \mathbb{Z}_+$. Let $a_n = n^k$, for all $n \in \mathbb{N}$. Then $\Delta^k a_n = k!$.

Example 3.4.3

Let $a_n = c^n$, for all $n \in \mathbb{N}$. Show that $\Delta a_n = (c-1)a_n$.

Solution

$\Delta a_n = c^{n+1} - c^n = c^n(c-1) = (c-1)a_n$.

Below are some properties of the difference operator.

Theorem 3.4.1: Difference operator is a Linear Operator

Let a_n and b_n be sequences, and let c be any number. Then

1. $\Delta(a_n + b_n) = \Delta a_n + \Delta b_n$, and
2. $\Delta(ca_n) = c\Delta a_n$.

Proof

Very simple calculation.

Definition: Falling Powers

The falling power is denoted by $x^{\underline{m}}$ and it is defined by $x(x-1)(x-2)(x-3)\cdots(x-m+1)$, with $x^{\underline{0}} = 1$.

Falling powers are useful in the difference calculus because of the following property:

Theorem 3.4.2

$\Delta n^{\underline{m}} = m\, n^{\underline{m-1}}$.

Proof

$$\begin{aligned}
\Delta n^{\underline{m}} &= (n+1)^{\underline{m}} - n^{\underline{m}} \\
&= (n+1)n(n-1)(n-2)(n-3)\cdots(n-m+2) - n(n-1)(n-2)(n-3)\cdots(n-m+1) \\
&= n(n-1)(n-2)(n-3)\cdots(n-m+2)((n+1)-(n-m+1)) \\
&= n(n-1)(n-2)(n-3)\cdots(n-m+2)\, m \\
&= m\, n^{\underline{m-1}}.
\end{aligned}$$

Let us now consider the sequence of numbers in the example 3.4.1.

Number of points on the circle	0	1	2	3	4	5	6	7	8
Number of regions	1	1	2	4	8	16	31	57	99
Δa_n		0	1	2	4	8	15	26	42
$\Delta^2 a_n$			1	1	2	4	7	11	16
$\Delta^3 a_n$				0	1	2	3	4	5
$\Delta^4 a_n$					1	1	1	1	1

Since the fourth difference is constant, a_n should be polynomial of degree 4. Let's explore how to find this polynomial.

Definition

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, k \leq n, n \in \mathbb{N} \cup \{0\}. \tag{3.4.1}$$

Theorem 3.4.2

$$\Delta\binom{n}{k} = \binom{n}{k-1}, k \leq n, n \in \mathbb{N} \cup \{0\}. \tag{3.4.2}$$

Proof

Note that

$$\binom{n}{k} = \frac{1}{k!} n^{\underline{k}}. \tag{3.4.3}$$

Consider $\Delta\binom{n}{k} = \Delta(\frac{1}{k!} n^{\underline{k}})$

$= \frac{1}{k!}\Delta(n^{\underline{k}})$

$= \frac{k}{k!} n^{\underline{(k-1)}}$

$= \frac{1}{(k-1)!} n^{\underline{(k-1)}}$

$= \binom{n}{k-1}$.

We can also see this by considering Pascal's triangle.

Theorem 3.4.3 Newton's formula

Let $a_0, a_1, a_2, \cdots$ be sequence of numbers such that $\Delta^{k+1} a_0 = 0$. Then the n th term of the original sequence is given by

$$a_n = \frac{1}{k!}(\Delta^k a_0) n^{\underline{k}} + \cdots + a_0 = a_0 + \binom{n}{0}\Delta a_1 + \binom{n}{2}\Delta^2 a_2 + \cdots + \binom{n}{k}\Delta^k a_k. \tag{3.4.4}$$

Proof

Exercise.

If n th term of the original sequence is linear then the first difference will be a constant. If n th term of the original sequence is quadratic then the second difference will be a constant. A cubic sequence has the third difference constant.

Source

- Thanks to Olivia Nannan for the diagram.
- Reference: Kunin, George B. "The finite difference calculus and applications to the interpolation of sequences." MIT Undergraduate Journal of Mathematics 232.2001 (2001): 101-9.
- Reference: Samson, D. (2006). Number patterns, cautionary tales and finite differences. *Learning and Teaching Mathematics, 2006*(3), 3-8.

3.E: Number Patterns (Exercises)

Exercise $3.E.1$: Hexagonal numbers (cornered)

Consider the hexagonal numbers are the sequence $1, 6, 15, 28, 45, 66 \cdots$ Predict the n th term. Explain your prediction.

Answer

$2n^2 - n$.

Exercise $3.E.2$: Finite sum

For each of the following, find the sum and explain your reasoning. Please do not use any formula.

1. $1+3+5+7+9+\cdots+197+199$
2. $1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^{16}}+\frac{1}{2^{17}}$

Answer

1. $1+3+5+7+9+\cdots+197+199$

Notice that $1, 3, 5, 7, \cdots$terms of a sequence. This is an Arithmetic Sequence because the difference remains the same between the terms throughout the entire sequence. Hence,$a = 1 \& d = 2$.

Consider,

$S_n = 1+3+5+7+9+\cdots+197+199$

$Sn = 199+197+195+193+191+\cdots+3+1$

By adding we get,

$(2S_n = 200+200+200+200+200+\cdots+200+200$

$2S_n = 100(200)$

$S_n = ((100)/2))(200)$

$S_n = (50)(200)$

$S_n = 10000$

Hence, the sum of the sequence is 10000.

2.

Exercise $3.E.3$: Proof by induction

Consider the sequence $4, 10, 16, 22, 28, , . . ,$ assume that the pattern continues.

1. Show that the n^{th} term of this sequence can be expressed as $6n - 2$.
2. Prove by using induction for all integers $n \geq 1, 4+10+16+\cdots+(6n-2) = n(3n+1)$

Answer

1.

Term	First difference
4	
10	6
16	6
22	6

Notice that the first difference is constant. Hence the n^{th} term is a linear function.

Let $t_n = a_n + b$.

Then we need to find a, b.

First Equation: Let $n = 1$

$t_1 = 4$

$4 = a(1) + b$

$4 = a + b$

Second Equation: Let $n = 2$

$t_2 = 10$

$10 = a(2) + b$

$10 = 2a + b$

To find a, we use $10 = 2a + b$ and $-4 = a + b$. Therfore, $6 = a$.

Now to find b, we use $a = 6$ and $4 = a + b$,

$4 = (6) + b$

$4 - 6 = b$

$-2 = b$.

Hence,$t_n = 6n - 2$.

2. Step 1: Base Step: Show that this statement is true for the smallest value

Check statement is true for n = 1.

L.H.S = 4

R.H.S = n(3n + 1)

= (1)(3(1) + 1)

= (1)(3 + 1)

= (1)(4)

= 4

Hence, the statement is true for n = 1.

Step 2: Induction Assumption:

We shall assume that the statement is true for n = k.

4 + 10 + 16 + . . . + (6k − 2) = k(3k + 1)

Step 3: Induction:

We shall show that the statement is true for n = k + 1.

4 + 10 + 16 + . . . + (6k − 2) + (6 (k + 1) − 2) = (k+1)(3(k + 1) + 1)

Consider, L.H.S = 4 + 10 + 16 + . . . + (6k − 2) + (6 (k + 1) − 2)

= k(3k + 1) + (6 (k + 1) - 2)

= k (3k + 1) + (6k + 6 - 2)

= k (3k + 1) + (6k + 4)

= 3k 2 + k + 6k + 4

= 3k 2 + 7k + 4

= (k + 1)(3k + 4)

Hence, the statement is true for n = k + 1

Therefore, by induction the statement is true, ∀n ∈ N.

Exercise $\mathbf{3.E.4}$: Proof by induction

Consider the sequence $3, 11, 19, 27, 35, \ldots,$ assume that the pattern continues.

1. Show that the n^{th} term of this sequence can be expressed as $8n - 5$.
2. Prove by using induction for all integers $n \geq 1, 3 + 11 + 19 \cdots + (8n - 5) = 4n^2 - n$.

Exercise $\mathbf{3.E.5}$: Tribonacci

Let's start with the numbers $0, 0, 1,$ and generate future numbers in our sequence by adding up the previous three numbers. Write out the first 15 terms in this sequence, starting with the first 1.

Exercise $\mathbf{3.E.6}$: Proof by induction

The sequence $b_0, b_1, b_2 \ldots.$ is defined as follows: $b_0 = 1, b_1 = 3, b_2 = 5,$ and for any integer $n \geq 3,\ b_n = 3b_{n-2} + 2b_{n-3}$.

1. Find b_3, b_4, b_5 and b_6.
2. Prove that $b_n < 2^{n+1}$ for all integers $n \geq 1$.

Exercise $\mathbf{3.E.7}$: Quadratic Sequence

Find the n^{th} term of the sequence $5, 10, 17, 26, 37, \cdots,$ assume that the pattern continues.

Answer

$(n+1)^2 + 1 = n^2 + 2n + 2$

Exercise $\mathbf{3.E.8}$: Proof by induction

Prove by using induction: for all integers $n \geq 1,\ 1 + 4 + 7 \cdots + (3n - 2) = \frac{n(3n-1)}{2}$.

Answer

Step 1: Base Step: Show that this statement is true for the smallest value

Check statement is true for n = 1.

L.H.S = 1

R.H.S = n(3n−1) / (2)

= (1)(3(1) − 1) / (2)

= (1)(3 − 1) / (2)

= (1)(2) / (2)

= 1

Hence, the statement is true for n = 1.

Step 2: Induction Assumption:

We shall assume that the statement is true for n = k.

1+4+7...+(3k−2)= k(3k−1) / (2)

Step 3: Induction:

We shall show that the statement is true for n = k + 1.

1+4+7...+ (3k - 2) + (3(k + 1) − 2) = (k + 1)(3(k + 1) − 1) / (2)

Consider, L.H.S = k(3k − 1) / (2) + (3 (k + 1) − 2)

= k (3k − 1) / (2) + (3k + 3) − 2)

= k (3k − 1) / (2) + (3k + 1)

= (3k 2 + k) / (2) + (3k + 1)

= (3k 2 + k + 3k + 1) / (2)

= (3k 2 + 4k + 1) / (2)

= ((k + 1)(3k + 1)) / (2)

Hence, the statement is true for n = k + 1

Therefore, by induction the statement is true, ∀n ∈ N.

Exercise $3.E.9$: Recognizing sequence

Predict n^{th} term of the sequence $\frac{2}{3}, \frac{3}{4}, \frac{4}{5} \cdots$ assume that the pattern continues. Explain your prediction.

Answer

$\frac{n}{n+1}$.

Exercise $3.E.10$: Recognizing sequence

Consider the sequence $t_1 = 1, t_2 = 3 + 5, t_3 = 7 + 9 + 11, \cdots$. Predict the n th term. Justify your prediction.

Exercise $3.E.11$: Proof by induction

Show that the perimeter of the design by joining n hexagons in a row is $8n + 4$ cm.

Exercise $3.E.13$: Pentagonal Numbers (cornered)

Find the n^{th} term of the sequence $1, 5, 12, 22, \cdots$;assume that the pattern continues.

Exercise $3.E.14$: Square Pyramidal numbers

Find the n^{th} term of the sequence $1, 5, 14, 30 \cdots$;, assume that the pattern continues.

Exercise $3.E.15$: Difference

Compute the difference of each of the following sequences:

1. $a_n = n^3$
2. $a_n = n^3$
3. $a_n = \binom{n}{3}$

Answer

1. $n^2 + 2n + 1$
2. $3n^2$
3. $\binom{n}{2}$.

CHAPTER OVERVIEW

4: Basic Concepts of Euclidean Geometry

Learning Objectives

Develop students':

- ability to visualize problems
- familiarity and facility with a wide range of geometry facts and problem-solving techniques
- understanding of the logical structure of geometry-axioms, conjectures, theorems and counterexamples
- visual reasoning

In Greek, "geo" means earth, and "metron" means measure. Egyptians were among the first people to use geometry to survey the land. The study of geometry was carried on by the Greeks Thales, Pythagoras (550 BC), Plato, and Euclid (father of geometry, 300 BC). They not only asked "how" and "what" but also asked, "why." At the foundations of any theory, there are truths, which are taken for granted and can't be proved or disproved. These are called axioms. The first axiomatic system was developed by Euclid in his books called "Elements". Research in teaching and learning of geometry has given strong support to the van Hiele theory. This theory was developed in the late 1950s by two Netherlands mathematics teachers. they observed that in learning geometry, students seem to progress through the following sequence: recognition, analysis, relationship, deduction (the study of geometry as a mathematical system), and axiomatic.

4.1: Euclidean geometry
4.2: 2-D Geometry
4.3: 3-D Geometry
4.4: Transformations
4.5: Symmetry
4.6: Summary
4.E: Basic Concepts of Euclidean Geometry (Exercises)

Thumbnail: This image illustrates in 3D a stereographic projection from the north pole onto a plane below the sphere. (CC BY-SA 4.0; Mark.Howison).

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4.1: Euclidean geometry

Euclidean geometry, sometimes called parabolic geometry, is a geometry that follows a set of propositions that are based on Euclid's five postulates.

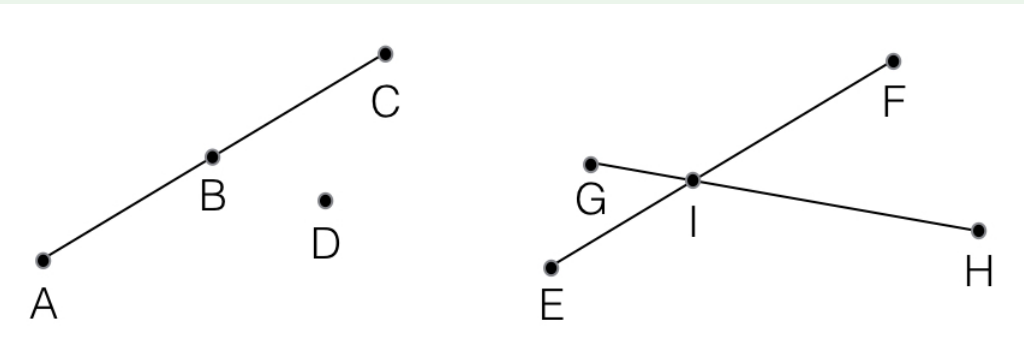
There are two types of Euclidean geometry: plane geometry, which is two-dimensional Euclidean geometry, and solid geometry, which is three-dimensional Euclidean geometry.

The most basic terms of geometry are a point, a line, and a plane. A point has no dimension (length or width), but it does have a location. A line is straight and extends infinitely in the opposite directions. A plane is a flat surface that extends indefinitely.

Points

Definitions:

Please refer to the image below for examples.



Collinear points: points that lie on the same straight line or line segment. Points A, B, and C are collinear.

Line Segment: a straight line with two endpoints. Lines AC, EF, and GH are line segments

Ray: a part of a straight line that contains a specific point. Any of the below line segments could be considered a ray

Intersection point: the point where two straight lines intersect, or cross. Point I is the intersection point for lines EF and GH.

Midpoint: a point in the exact middle of a given straight line segment. Point B is the midpoint of line AC.

Think Out Loud

How many different lines can you draw through a fixed point? How many different lines can you draw through two fixed points? How many different lines can you draw through three fixed points?

Angles

Definitions:

Angle: $\measuredangle ACB$. Normally, Angle is measured in degrees (^{0}) or in radians rad).

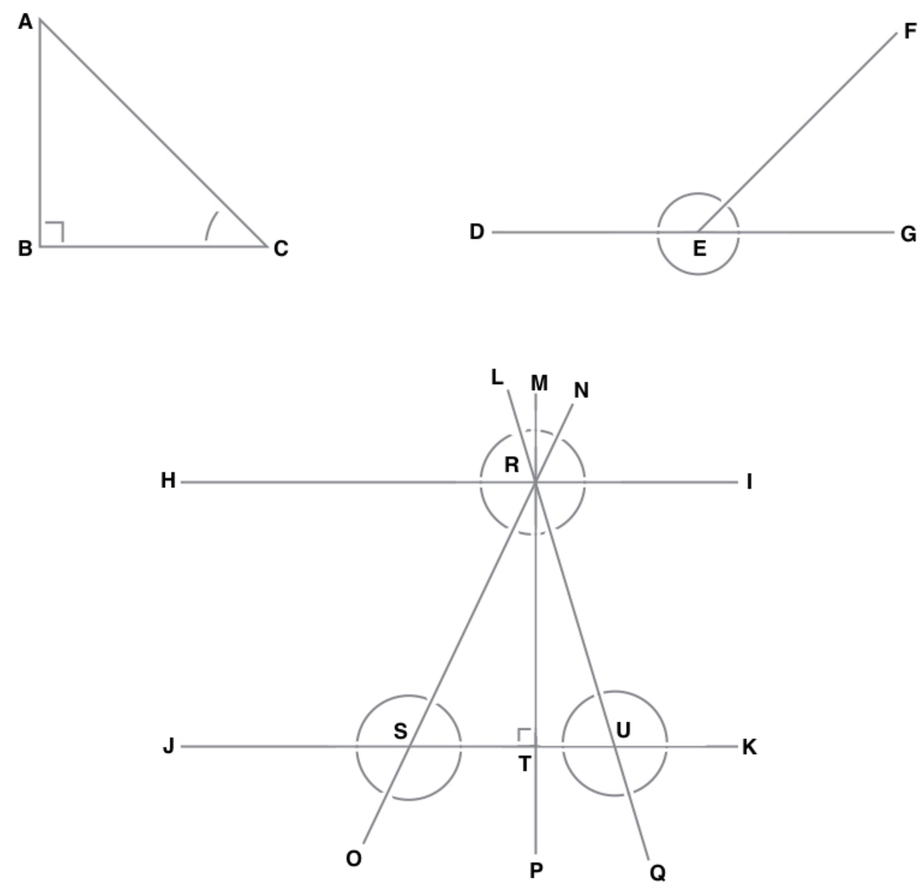
Right angle: Angles which measure 90° - $\measuredangle ABC$

Obtuse angle: Angles which measure > 90° - $\measuredangle CDE$

Acute angle: Angles which measure < 90° - $\measuredangle FDE$

Straight angle: Angles which measure 180° $\measuredangle CDF$

Reflex angle: A reflex angle is an angle, that is measured > 180°, which adds to an angle to make 360° - $\measuredangle CDE$'s reflex angle is $\measuredangle CDF + \measuredangle FDE$



Adjacent angles: Have the same vertex and share a side. $\measuredangle HRL$, $\measuredangle HRO$ are adjacent.

Complementary angles: add up to 90°. $\measuredangle PRQ$, $\measuredangle QRI$ are complementary angles.

Supplementary angles: add up to 180°. $\measuredangle JSN$, $\measuredangle NSK$ are supplementary angles.

Vertical angles (X property): Angles which share line segments and vertexes are equivalent. $\measuredangle JSR$, $\measuredangle OST$ are vertical angles. They share the same degree value.

Corresponding angles (F property): Angles which share a line segment that intersects with parallel lines, and are in the same relative position on each respective parallel line, are equivalent. $\measuredangle IRQ$, $\measuredangle KUQ$ are corresponding angles. They share the same degree value.

Alternate interior angles (Z property): Angles which share a line segment that intersects with parallel lines, and are in opposite relative positions on each respective parallel line, are equivalent. $\measuredangle HRS$, $\measuredangle RST$ are alternate interior angles. They share the same degree value.

Bisecting an Angle: To bisect an angle is to draw a line **concurrent line** through the angle's vertex which splits the angle exactly in half. This is possible using a compass and an unmarked straightedge.

Trisecting an Angle: To trisect an angle is to use the same procedure as bisecting an angle, but to use two lines and split the angle exactly in thirds. This is an ancient impossibility - it is impossible to accomplish using a compass and an unmarked straightedge.

Lines

Definitions:

Parallel lines: Lines which, drawn on a 2-dimensional plane, may extend forever in either direction without ever intersecting. Lines HI and JK are parallel.

Perpendicular lines: Lines which intersect at exactly a 90° angle. Lines HI and MP are perpendicular.

Concurrent lines: Lines that all intersect at the same point. Lines HI, LQ, MP, $NO$are concurrent.

Skew lines: Lines which, drawn in a 3-dimensional space, are both neither parallel nor perpendicular and do not intersect.

Perpendicular Bisector: A line that is perpendicular to a given line and bisects it is called a perpendicular bisector.

Example 4.1.1:

Draw a cube and connect all the edges. Can you find two skew lines?

Planes

Definition:

A **plane** is a two-dimensional space that extends infinitely in all directions. For example, the graph of functions takes place on a **Cartesian plane** or a plane with coordinates. The plane continues in both the x and, y directions.

Axioms

The points, lines, and planes are objects with the relations given by the following axioms:

1. There is a unique line passing through two distinct points.
2. If two points lie in a plane, then any plane containing those points lies in the plane.
3. There is a unique plane containing three non-collinear points.
4. If two planes meet, then their intersection is a line.

Euclid's Five P**ostulate**s

Euclid's Five Postulates

Euclid's five postulates can be stated as follows

1. It is possible to draw a straight line segment joining any two points.
2. It is possible to indefinitely extend any straight line segment continuously in a straight line.
3. Given any straight line segment, it is possible to draw a **circle** having the segment as a radius and one endpoint as its center.
4. All right angles are equal to each other or congruent
5. Through a given point not on a given straight line, only one line can be drawn parallel to a given line.

4.2: 2-D Geometry

Two-dimensional figures

Definition:

Vertices (pl.) or vertex (sg.): a point or corner which joins two edges of a shape.

Edges: Line which describes one of the outer borders of a shape.

A curve is called a **closed curve** if we can trace the figure in such a way that our starting point and ending point are the same.

A **simple closed curve** is a curve that we can trace without going over any point more than once while starting point and ending point are the same.

Polygons

A polygon is a closed, 2-dimensional shape, with edges(sides) being straight lines. The word “polygon” is derived from Greek for “many angles”. The names of the polygons are taken from the Greek number prefixes followed by –gon, with only a couple exceptions.

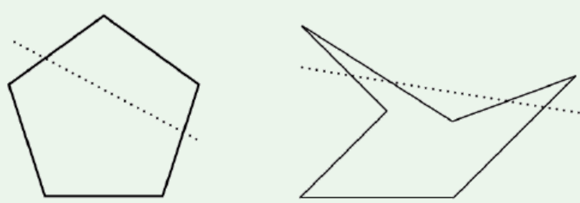
Example 4.2.1:

3-tri 4-tetra 5-penta 6-hexa 7-hepta 8-octa 9-ennia or nona 10-deca

Definitions:

Convex polygons are polygons which, if a line is drawn across them at any point, only two edges will be intersected by the line.

Concave polygons are those in which more than two edges are intersected by a line drawn across the shape.



A polygon is considered to be **regular** if all of its sides have the same measure. By definition, this also implies that all of the polygon's internal angles will be the same.

A **diagonal** is a line that joins any two vertices of a shape that are not adjacent to each other.

Two polygons are **Congruent** if they are identical in size and shape.

- **Interior angles** of a polygon are the angles between two adjacent sides on the inside of the polygon
- **Exterior angles** of a polygon are the angles between two adjacent sides on the outside of the polygon
- **Central angles** are angles between two points on a polygon, measured from the center of the polygon. In circles, this measure is given in radians or degrees.

The **center of a regular polygon** is a point that is the same distance (equidistant) from all of its vertices. An irregular polygon will not have a center.

The **radius** of a circle is given by the distance from the centre to the edge.

Regular Polygons

Properties:

A regular polygon's **apothem** is a line that is drawn from its center to the midpoint of one of its faces

A regular polygon's **radius** is a line that is drawn from its center to one of its vertices

The **perimeter** of a shape is the sum of the length of all its faces

The **area** of a polygon is the measure of its surface area, given in square units such as cm^2.

Triangles

A triangle is a polygon with three edges and three vertices and can be defined by a side and two angles. This means that if we know the length of one side and the measure of two angles in a triangle, we can know the measures of all three sides and angles.

Definition:

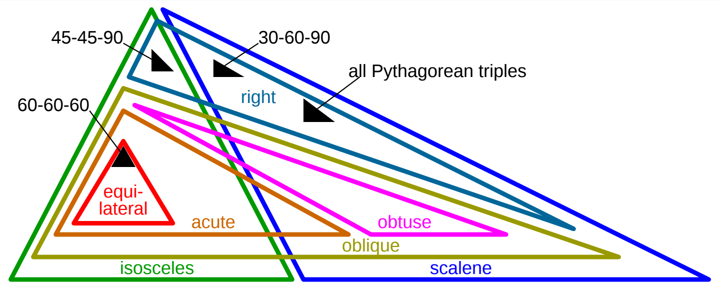
- **Equilateral triangle**: A triangle with all sides equal in length.
- **Isosceles triangle**: A triangle with two sides equal in length.
- **Scalene triangle**: A triangle with no sides equal in length.
- **Right triangle**: A triangle with one interior angle that measures 90°.
- **Acute-angled triangle: All** angles are acute (between 0° and 90°).
- **Right-angled triangle: One** angle is a right angle (90°).
- **Obtuse-angled triangle: One** angle is obtuse (between 90° and 180°).

Example 4.2.2: Classify Triangles

Complete the following table by sketching a triangle, if possible. Otherwise, justify your answer.

	Isosceles Triangle	Equilateral Triangle	Scalene Triangle
Acute angled			
Obtuse-angled			
Right-angled			

Would you be able to sketch all of the above triangles inside a rectangle? Here is an Euler diagram of types of triangles:



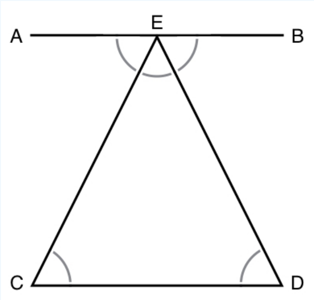
Special properties of triangles

Definition:

- **Angle Bisector:** A line that bisects an angle of a triangle.
- **Median:** A line segment that connects a vertex to the midpoint of the opposite side.
- **Altitude:** A perpendicular line segment that connects a vertex to the side opposite to that vertex.

Example 4.2.3: Proof - The Sum of Interior Angles of Any Triangle

The sum of the interior angles of any triangle is 180°:



Lines AB and CD are parallel.

By the alternate interior angles property, $\measuredangle AEC$ and $\measuredangle ECD$ are equivalent.

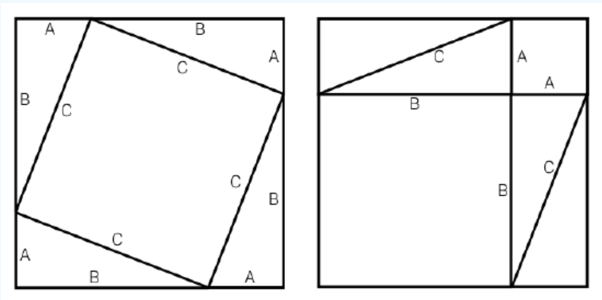
By the alternate interior angles property, $\measuredangle BED$ and $\measuredangle EDC$ are also equivalent.

Since $\measuredangle AEC + \measuredangle CED + \measuredangle BED = 180^\circ$,

Then $\measuredangle ECD + \measuredangle CED + \measuredangle EDC = 180^\circ$.

Example 4.2.4: Proof - $a^2 + b^2 = c^2$, or the Pythagorean Theorem

Method 1: Let us draw two squares made by using four right triangles with sides A, B, C, where A is the vertical side, B is the horizontal side, and C is the hypotenuse:



Since both squares are the same size, we can say that their areas are equivalent.

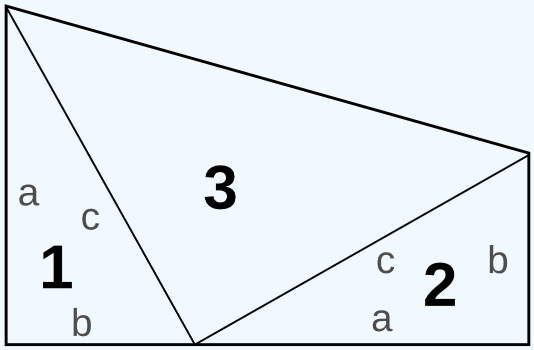
Both squares also contain four right triangles.

When we remove the triangles from both squares, we are left with c^2 on the left, and $a^2 + b^2$ on the right.

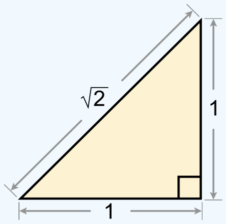
Remember, we have two squares of equal area, and we removed the same quantity of area from each.

So: $a^2 + b^2 = c^2$.

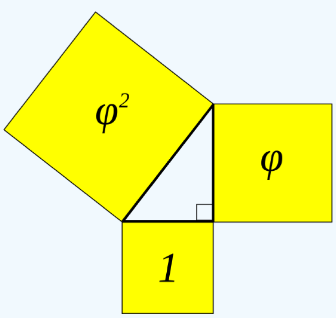
Method 2: President Garfield method



Example 4.2.5: $\sqrt{2}$ is Irrational

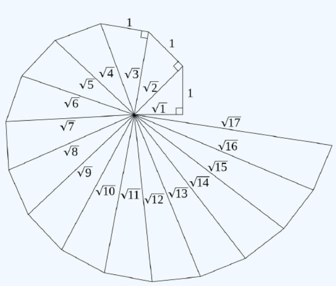


Example 4.2.6: Kepler Triangle

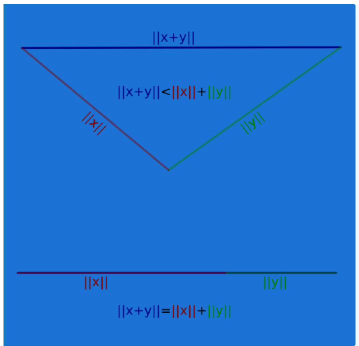


A Kepler Triangle is a right angle triangle with sides $1, \sqrt{\phi}, \phi$, where ϕ is the golden ratio.

Example 4.2.7:



Triangle inequality



$$x + y \geq z\,.$$

The sum of the lengths of two sides of any triangle is always greater than or equal to the length of the third side.

Definition:

Triangles are said to be **similar triangles** if their internal angles are the same, but their sides are different sizes. The sides or similar triangles are in relative proportion: the ratio by which one is bigger or smaller than the other is the same for all sides.

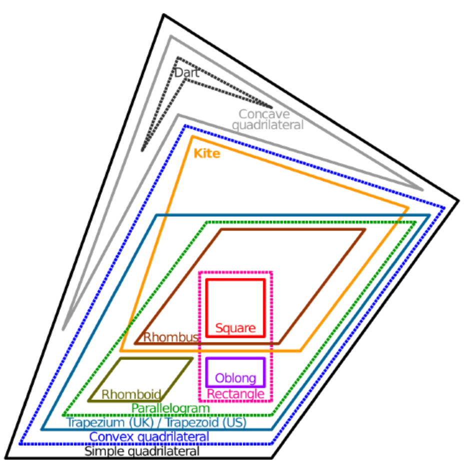
Triangles are **congruent** if both their sides and their angles are the same.

Quadrilaterals

Quadrilaterals are polygons with four edges and four vertices.

- A **kite** is a quadrilateral with adjacent pairs of sides that are equal.
- A trapezoid is a quadrilateral with one pair of parallel sides.
- A **parallelogram** is a quadrilateral with opposite sides that are parallel and equal.
- A rhombus is a parallelogram that has all four sides equal.
- A rectangle has four right angles, and thus the sides are parallel and equal in pairs (a quadrilateral with 4 square corners).
- A square has four right angles and four equal sides. (A quadrilateral with 4 square corners and 4 equal sides. Thus, a square is a special type of rectangle!)

Here is an Euler diagram of types of quadrilaterals:

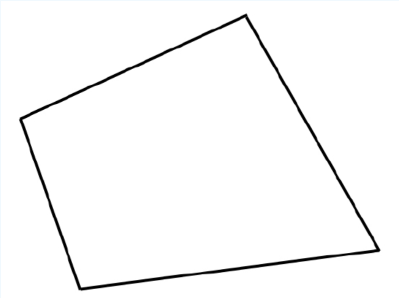


Thinking Out Loud:

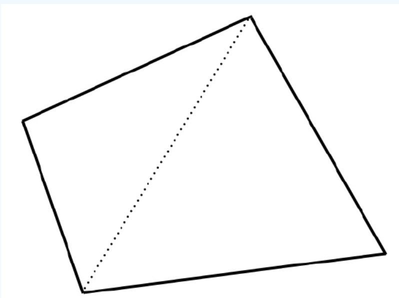
Draw a quadrilateral. Find the midpoint of each side (paper folding might help). Connect the midpoints of the sides which have a common vertex. What shape have you created? Is this true for any quadrilateral?

Example $4.2.4$: Proof - The sum of the interior angles of a quadrilateral is 360°

First, let's draw a quadrilateral:



Then, we will make this shape into an aggregate of shapes we know more about: triangles.



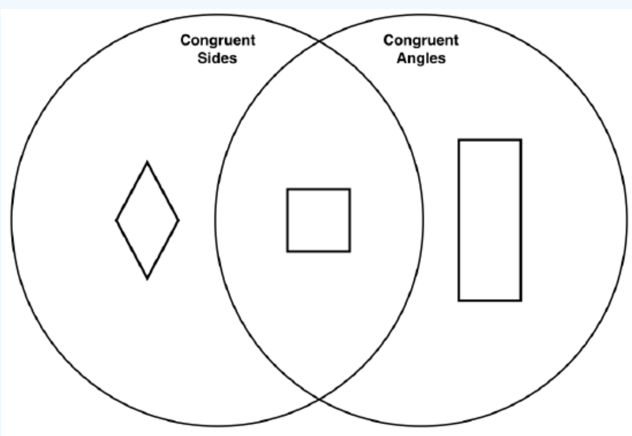
As we can see, a quadrilateral is made of two triangles. Since we know that the sum of the internal angles of each triangle is 180°, we can add those two together to find that the sum of the internal angles of a quadrilateral is 360°.

Thinking Out Loud:

Does the proof of the interior angles of a quadrilateral hold when the quadrilateral is concave? Why? Why not?

Example $4.2.8$:

Express the relationship between a rhombus, square, and rectangle with a Venn diagram.

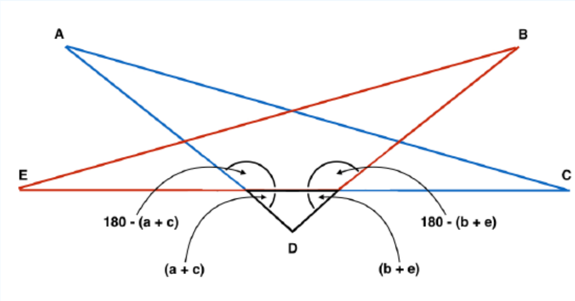


Pentagrams

Pentagrams are polygons that make a five-pointed star.

Example 4.2.9: Proof: The sum of Vertex Angle of a Pentagram is 180°

Consider the following pentagram:



As we can see, we have determined the interior angles of the black triangle in the general case, using the internal vertex angles from the blue and red triangles.

Since the sum of a triangle's interior angles is 180°,

$a+b+c+d+e=180^{\circ}$.

Other Polygons

The sum of the interior angles of polygons with n sides are given by the formula $(n-2)180^{\circ}$. This can be shown by the fact that, n sided polygons can be divided into $(n-2)$ triangles.

Example 4.2.10:

Consider a regular decagon: $n=10$. What is the sum of its internal angles?

Since the sum of interior angles $=(n-2)180^{\circ}$,

Then the sum of a regular decagon's interior angles $=(10-2)180^{\circ}$

$=(8)180^{\circ}$

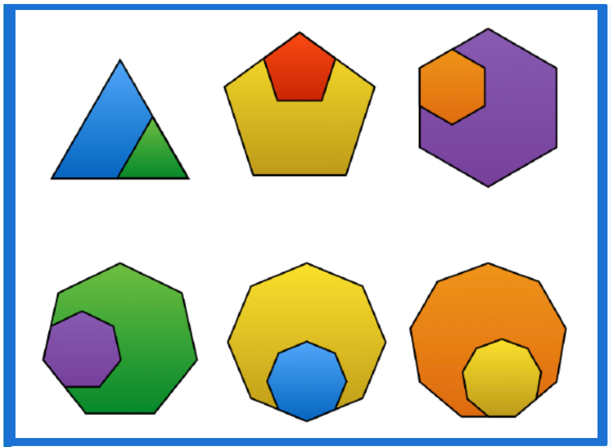
$=1440^{\circ}$

Similarity

Definition

Polygons with n sides are similar when they have an equal number of sides that are proportional to each other in the same ratio.

Example 4.2.8:



Tessellation

Definition

Tessellation refers to the "tiling" of polygons, as in ceramic tiles or quilts. There are regular tessellation and semi-regular tessellation. A regular tessellation is made up of regular polygons that are all of the same shape and all meet at a common vertex. There are only three types of regular tessellations which are comprised of equilateral triangles, squares and hexagons. The sum of the angles of polygons in regular tessellations forms 360 degrees around each vertex.

Thinking Out Loud:

Given a 2-dimensional stage, which shapes can be tiled? Which can't? Why?

Given a 3-dimensional space, does your answer change? Why or why not?

Definition:

A semi-regular tessellation is made up of two or more types of regular polygon. Polygon arranged in the same cyclic order. There are eight semi-regular tessellations which comprise different combinations of equilateral triangles, squares, hexagons, octagons and dodecagons.

Example 4.2.11: Semi-regular tessellations.

Definition:

Non-Regular Tessellations

Non-Regular Tessellations are those in which there are no restrictions regarding the shapes used or their arrangement around vertices. It is believed there is an infinite number of irregular tessellations.

Curved figures

- Circles
- Semicircle
- Spiral
- Parabola
- Ellipse
- Hyperbola
- Crescent

Circle Geometry

Definitions:

The **center of a circle** is a point in the circle that is equidistant from all points along the circle's edge. This distance is given by the radius.

The **radius of a circle** is the distance from the center of the circle to the edge of the circle.

The **diameter of a circle** is the distance across a circle at its widest point and is twice the circle's radius.

The **circumference** of a circle is the distance around the circle.

A **chord** is a line that joins two points on the circumference of a circle.

An **arc** is a part of the circumference of a circle.

A **tangent line** meets a circle at one point only.

An **inscribed angle** is an angle given by two chords that share a common endpoint.

A **central angle** is an angle describing an arc. This is measured, in circles, in radians.

Properties

1. The perpendicular line from the center of a circle to a chord bisects the chord.
2. $\pi = \frac{Circumference}{Diameter}$.
3. $A = \pi r^2$
4. The angle between the tangent at a point, and the radius from the center to that point is 90°.
5. The measure of a central angle is twice as large as the measure of an inscribed angle that cuts the same arc of a circle.
6. The angle in a semicircle is a right angle.

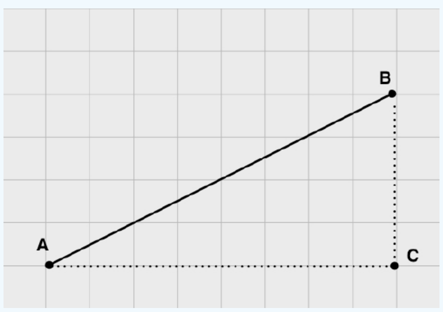
Thinking Out Loud:

Can we inscribe regular polygons in a paper cut circle?

Coordinate Geometry

Coordinate geometry applies the Cartesian plane to the geometry we have already learned. Each vertex of a shape now is given an ordered Cartesian pair $(x,\ y)$ to give its position on the grid. To find distances between points, we create the right triangles and apply the Pythagorean Theorem.

Example 4.2.10:



In a general case, point A has coordinates $(x_1,\ y_1)$, and point B has coordinates $(x_2,\ y_2)$. Point C, the point we use to create a right triangle, has coordinates $(x_2,\ y_1)$.

Let $A(x_1,\ y_1)$, and $B(x_2,\ y_2)$ be points.

Midpoint Coordinates The midpoint coordinates of a line can be thought of as the average of both A and B coordinates from both endpoints. So, the formula for the midpoint of a line is:

$$\left(\frac{x_1+x_2}{2},\ \frac{y_1+y_2}{2}\right)$$

Slope of a Line:

The slope of a line, or how much it rises for a given amount of horizontal travel, can be thought of as $m = \frac{rise}{run} = \frac{\Delta y}{\Delta x}$.

Δ is the mathematical symbol indicating the change in a quantity.

Hence the slope of the line AB is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$.

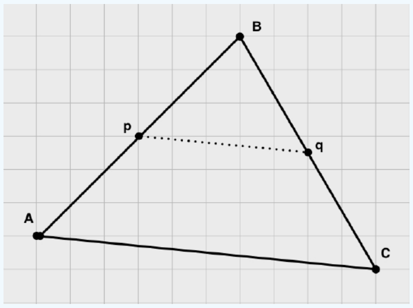
Distance Formula

The distance between the points A and B is given by $\sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$.

Example 4.2.12:

Show, by using coordinate geometry, that the line connecting the midpoints of two sides of a triangle is parallel to the other side.

First, let's draw a picture:



Let us call the midpoints of lines AB and BC p and q, respectively.

Consider the slope of line AC:

$$m_{AC}= \frac{y_c - y_a}{x_c - x_a}$$

Consider point p, the midpoint of line AC:

$$p = \left(\frac{x_a + x_b}{2}, \frac{y_a + y_b}{2} \right)$$

Consider point q, the midpoint of line CB:

$$q = \left(\frac{x_c + x_b}{2}, \frac{y_c + y_b}{2} \right)$$

Consider the slope of line pq

$$m_{pq}= \left(\frac{y_q - y_p}{x_q - x_p} \right)$$

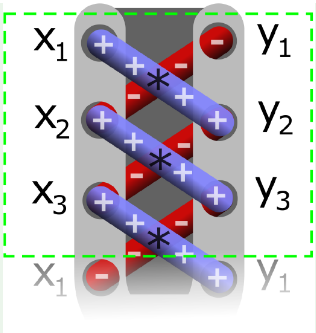
$$= \left(\frac{\left(\frac{y_c + y_b}{2} \right) - \left(\frac{y_a + y_b}{2} \right)}{\left(\frac{x_c + x_b}{2} \right) - \left(\frac{x_a + x_b}{2} \right)} \right)$$

$$= \left(\frac{(y_c + y_b) - (y_a + y_b)}{(x_c + x_b) - (x_a + x_b)} \right)$$

$$= \frac{y_c - y_a}{x_c - x_a}$$

Thus, we have shown that $m_{AC} = m_{pq}$. QED.

Shoelace formula:



Let $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$ be coordinates of the vertices of a polygon with n sides. Then the area of the polygon given by

$$A = \frac{1}{2}|x_1y_2 + x_2y_3 + \cdots + x_ny_1 - x_2y_1 - x_3y_2 - \cdots - x_1y_n| \tag{4.2.1}$$

Notice the blue laces and red laces in the attached figure.

Example 4.2.13:

Find the area of the triangle with vertices $(1, 2)$, $(2, 3)$ and $(3, 5)$. Make a shoelace diagram first!

By using the shoelace formula,

The area $A = \frac{1}{2}|(1)(3) + (2)(5) + (3)(2) - (2)(2) - (3)(3) - (1)(5)| = \frac{1}{2}$.

Ruler and Compass Constructions

Compass: A tool for marking a circle.

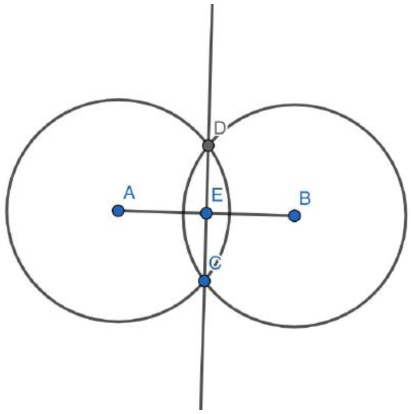
Straightedge: A ruler without any marks on it.

Geometric constructions are delightful by nature, they offer a playfulness to mathematics as well as a new form of creativity. But how do these constructions work, and what are some of the methodologies or proofs that have emerged throughout history? Let's uncover a few, To begin, we consider the creation of a bisector to a given line.

Construction 1: Bisecting a line

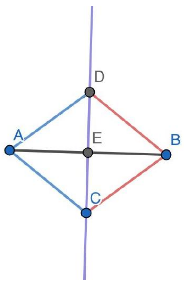
Say we are given a line AB.

Our first step is to choose point A as the center point for our compass. Extend the compass to create a circle with a radius a little larger than half of length AB and draw the circle. From the other point B, create a circle with the same radius this time with B as the center point. The 2 points of intersection of the two circles will be points on
the straight line CD that bisects AB.



But how can we prove that this line CD is actually a bisector of line AB? One method is to use congruent triangles to show geometrically that CD must be a bisector of AB.

If we join the points the following way, one sees that we've created two isosceles triangles DAC and DBC. Because the circles created on points A and B have the same radius, we know that points D and C are an equal distance away from points A and B, thus DB = BC = CA = AD. This demonstrates that the two triangles DAC and DBC are isosceles.



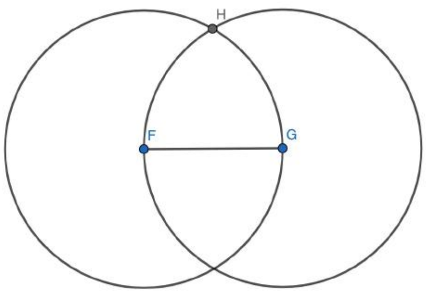
Both triangles also share the same base DC. We know that congruent triangles must have all the same angles and side lengths as each other since both triangles DAC and DBC have two side lengths equal to DB and one shared side length DC, they both have three identical sides to each other and are thus congruent. Congruent triangles have the same height as each other, thus we know that AE and EB are the same lengths which proves that the line we've created is indeed a bisector of line AB.

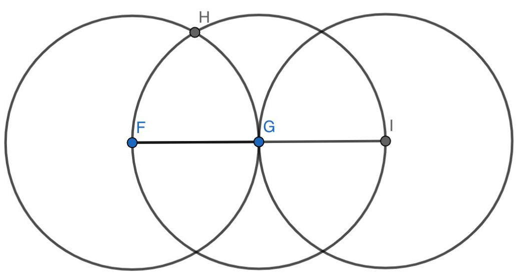
Construction 2: Creating a hexagon, given a line

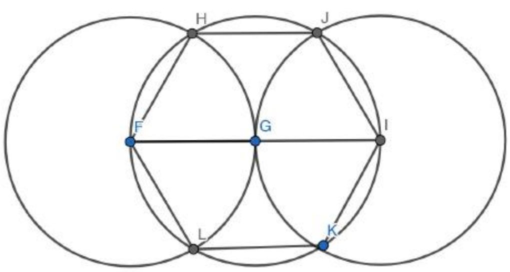
Given a line FG, create a hexagon with side length FG.

Using a compass, draw a circle with radius FG and center F. Create another circle with radius FG and center point G. Extend line FG to create a new line FI length 2(FG).

Construct yet another circle with radius FG and center point, I. Mark, each point of intersection amongst each of the three circles. Using points F, H, J, I, K, L as vertices, construct a hexagon.







A regular hexagon can be constructed from six congruent equilateral triangles. The use of circles of equal radius enables the construction of multiple equilateral triangles that in turn form a hexagon. For example, we see that triangle FHG is equilateral. Point H is of equal distance from F as is it from G, and the distance of HF and HG is equal also to the distance of FG. Therefore FHG must be equilateral. It then follows that FLG maintains the same criteria and is an equilateral triangle congruent to FHG. Furthermore, it follows that the pattern of construction around point I will also result in congruent equilateral triangles GJI and GKI, for circles of the same radius on I and G will form equilateral triangles much the same way that circles of the same radius formed equilateral triangles on F and G. Given that the angles FGH and IGJ sum to an angle of 120 degrees, it follows that supplementary angle HGJ will be 60 degrees. We know that lines GH and GJ are equal and congruent to lines FG and IF thus HGJ is an equilateral triangle congruent to FHG. Near identical processing will construct triangle LGK and we will see a constructed hexagon with vertices F, H, J, I, K, L. Alternate solutions to the construction of a regular hexagon with side length FG might include bisecting various angles, extending lines, creating identical angles, and many other techniques. The potential solutions for this problem, as with many others, are infinite, so long as the problem is actually solvable that is.

Activity

Construct a regular dodecagon using only a compass and a straight edge.

Hint

Start by drawing a circle with a center, let's say O. Then pick any point on the circle, let's say A. Now construct a circle with center A and the same radius as previous. Continue to construct seven circles using the intersecting points.

The following app/game may help in the construction:

https://www.euclidea.xyz/en/game/packs

For hundreds of years, there were three problems that endlessly tormented mathematicians. Many mathematicians strove endlessly to find solutions to the problems, now
known as the “three classical problems.”

Example 4.2.14:

Using an unmarked straight edge and a compass, try to do the following exercises:

- Square a circle - given a circle of a certain area, create a square with the same area.
- Trisect an Angle - divide an angle into three equal parts.
- Construct a cube with twice the volume of a given cube.

Source:

By Job Bouwman [CC BY-SA 4.0 (https://creativecommons.org/licenses/by-sa/4.0)], from Wikimedia CommonsBy derivative work: Pbroks13 (talk) Square root of 2 triangle.png: en:User:Fredrik Simplified drawing: Rubber Duck (☮ • ✍) (Square root of 2 triangle.png) [Public domain], via Wikimedia Commons

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Tags recommended by the template: article:topic

4.3: 3-D Geometry

Thinking Out Loud

When you slice an orange, what type of shape(s) can occur?

Polyhedra

Definition: Polyhedra

Polyhedra (pl.) are simple closed surfaces that are composed of polygonal regions.

A **polyhedron** (sg.) has a number of:

- **Vertices** - corners where various edges and polygonal corners meet
- **Edges** - lines where two polygonal edges meet
- **Faces** - the proper name for polygonal regions which compose a polyhedron

Polyhedra may be:

- **Convex** - shapes that follow the convex property of 2-dimensional geometry in 3-dimensional space.
- **Concave** - shapes that follow the concave property of 2-dimensional geometry in 3-dimensional space.

Nets are used when constructing polyhedra out of a single, contiguous, piece of material. The various polygons are laid out together, edges touching, to be cut out and folded together.

Polyhedra are said to be **prisms** if they have bases which are congruent and parallel polygons

Pyramids

Definition: Pyramids

Pyramids are created by joining a polygonal base to a point above it, called an **apex**. Each edge of the base, when joined by its vertices to the apex, creates a series of congruent triangles around the shape.

The more edges the base has, the more a pyramid approximates a cone.

Thinking Out Loud:

The sphere is the most symmetrical solid in space. Building a sphere isn't easy, so what other solids might we construct to approximate its symmetry? In order to construct a solid with lots of symmetry, we suppose our solid has flat sides and straight edges. What properties would such solids have in order to be symmetrical as possible?

Regular Polyhedra

Definition: Regular polyhedra (Platonic solids)

A polyhedron is said to be **regular** if:

- All of its faces are congruent
- All of its vertices join the same number of edges
- All of its edges join only two faces

Regular polyhedra are also called **the Platonic solids.**

Thinking Out Loud:

How many regular polyhedra are possible? How can you prove it?

ACTIVITY

Activity: Appreciate polygons and support the idea that there are exactly 5 platonic solids.

Suppose we want to tape regular n-gons together to make 3-dimensional shapes. We can make a cube, for example, by taping squares together. What are our options? We don't want to bend or fold the n-gons. Let's just concentrate on the corners of these objects.

Fact: To make a corner we'll need at least 3 regular n-gons.

Try making corners out of 3 n-gons. Which ones will work? Justify your conclusions.

Now try using four n-gons to make corners. Which ones will work? Justify your conclusions.

What about using five n-gons? Justify your conclusions.

Can we make corners out of six or more n-gons? Justify your conclusions.

A platonic solid is a 3-dimensional object made by taping together regular n-gons in such a way that each corner is the same, and has the same number of n-gons around it. Using the data you've gathered, please complete the following statement:

I have found that there are ____________ ways to tape regular n-gons together to make the corners of a platonic solid. Therefore, there are at most __________ platonic solids.

Video Demonstration can be found here.

Tetrahedron

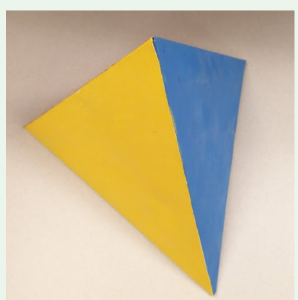
Definition: tetrahedron

A tetrahedron is a Platonic solid with:

- 4 faces
- 4 vertices
- 6 edges

The tetrahedron is bounded by four equilateral triangles and has the smallest volume for its surface area of the Platonic solids.

In Ancient Greece, a tetrahedron represents the property of dryness and corresponds to the element of Fire.



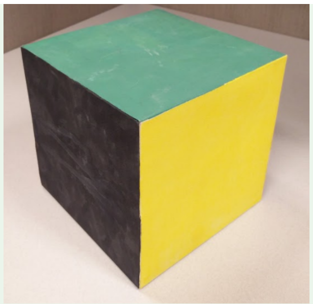
Cube (hexahedron)

Definition: Cube

A cube, or hexahedron, is a Platonic solid with:

- 6 faces
- 8 vertices
- 12 edges

The cube is bounded by six squares.



In Ancient Greece, the cube, standing firmly on its base, corresponds to the element of Earth.

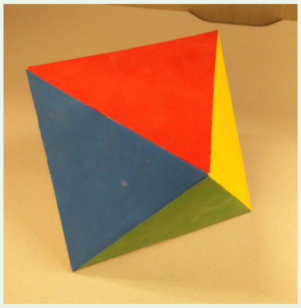
Octahedron

Definition: Octahedron

An octahedron is a Platonic solid with:

- 8 faces
- 6 vertices
- 12 edges

The octahedron is bounded by eight equilateral triangles. It rotates freely when held by two opposite vertices.



In Ancient Greece, the octahedron corresponds to the element Air.

Dodecahedron

Definition: Dodecahedron

A dodecahedron is a Platonic solid with:

- 12 faces
- 20 vertices
- 30 edges

The dodecahedron is bounded by twelve congruent regular pentagons.

In Ancient Greece, the dodecahedron corresponds to the universe because the zodiac has twelve signs corresponding to the twelve faces of the dodecahedron.

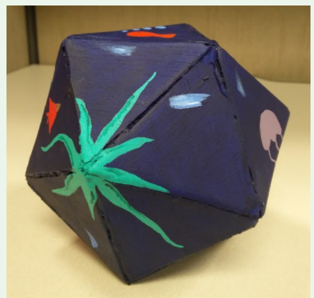
Icosahedron

Definition: Icosahedron

An icosahedron is a Platonic solid with:

- 20 faces
- 12 vertices
- 30 edges

The icosahedron is bounded by twenty equilateral triangles and has the largest volume for its surface area of the Platonic solids.



In Ancient Greece, the icosahedron represents the property of wetness and corresponds to the element of Water.

Thinking Out Loud:

Where in your life might you have seen the Platonic solids? How might the Platonic solids be useful?

ACTIVITY

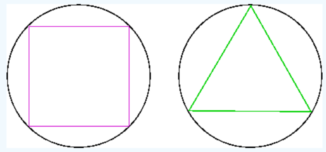
Activity: Holiday decorations

- **Materials/Equipment:**
- Card stock, Bristol board, used greeting cards or some other stuff but not too thick paper. You'll need lots of it for a class.
    - White glue.

CONSTRUCTION DIRECTIONS:

Step 1: Fix a radius on your circle cutter and mass produce a lot of circles. For example, if students were making cubes they'd need 6 per cube, if they make the icosahedrons they'll need 20 circles (in my opinion icosahedrons with flaps are the nicest to make!) Keeping the radius fairly small (eg 4 inches on my non-metric circle cutter) would let you get away with using up less paper and makes a nice sized decoration. Produce some extra circles for errors and a geometry discussion with students.

Step 2: Take one of the circles and inscribe the template square (for cubes) or equilateral triangle (for icosahedrons). (Fold to get the center of the circle and for the square the folds locate the vertices for the square. For the triangle you can pull out a protractor and measure off 3 120 degree angles to get the vertices.) Cut out the template shape.



Step 3: Trace this shape and trace it out on as many circles as you need. Work fast – slight imperfections are ok. Now take a rule and a nail (or something else a little sharp) and quickly trace over the lines to break the surface of the cardstock. Fold the sides up to get little 'cups' with a square or triangular base. I have gone through steps one to three and made over 120 circles folded into cups in one hour.

Step 4: Bring nets of the cube and icosahedrons (or pictures from the websites mentioned) to help guide the students in putting the shape together. Students will be gluing the flaps together to form the cube or icosahedrons. You have the option of gluing the flaps out or flaps in. The icosahedrons look really neat with the flaps out. Given them the white glue and lead them through putting it together. Again, make one yourself in advance so you can get a feel for how much guidance your students will need and if they need an older helper to finish it off.

In fact. you can also construct other polyhedra (the Archimedean solids for example) using this technique. I'd need the Pythagorean theorem to construct a cuboctahedron.

Video Demonstration can be found here.

References:

- The Mathlab.com, Making the Dice of the Gods, http://www.themathlab.com/wonders/godsdice/godsdice.htm (as of Oct. 6, 2012)
  - Aunt Annie's Crafts, Platonic solids, http://www.auntannie.com/Geometric/PlatonicSolids/ (as if Oct. 6, 2012).
- La Haye, Roberta. "Geometry and Art with a Circle Cutter." *Proceedings of Bridges 2012: Mathematics, Music, Art, Architecture, Culture*. 2012.

The article can be found in https://archive.bridgesmathart.org/2...s2012-425.html

Euler's Formula

Definition:Euler's Formula

There is a relationship between the number of faces (F), vertices (V), and edges (E) in any convex polyhedron, and knowing this relationship enables us to construct a formula that connects the number of faces, vertices, and edges.

Euler's formula for convex polyhedra is: $V+F=E+2$

That is: for any convex polyhedron, the number of vertices added to the number of faces is equal to two more than the number of edges.

Example 4.3.1:

Consider the dodecahedron:

$V=20,\ F=12,\ E=30.$

Let's see if Euler's formula holds:

$V+F=E+2$

$V+F=(20)+(12)=32$ and $E+2=(30)+2=32$

Excellent! It works.

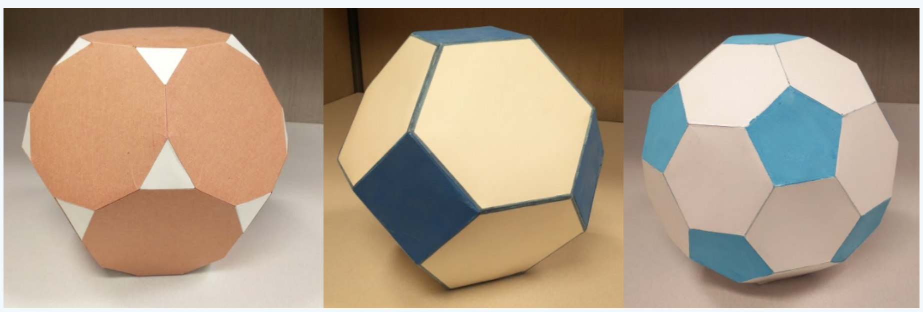
Truncated Regular Polyhedra

Definition:

Truncated regular polyhedra, which are also sometimes called **Archimedean solids**, must:

- Be composed of regular polygons
- Have identical vertices
- Not be a Platonic solid, prism, or anti-prism.

Example 4.3.2:



Non-Convex Uniform Polyhedra (Kepler-Poinsot Solids)

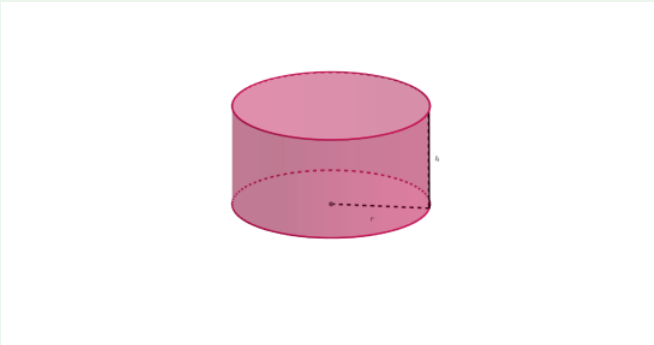
It is possible to construct regular polyhedra that are not convex - that is, shapes that have identical faces but also have incuts or void spaces. These solids are sometimes called Kepler-Poinsot polyhedra. These shapes can be made by building a regular dodecahedron or icosahedron and adding pyramidal or pentagramal volumes to each face. **These polyhedra do not always satisfy the Euler relation as it relates to Platonic solids.**

There are four Kepler-Poinsot polyhedra: three based on the dodecahedron and one built upon the icosahedron.

Example 4.3.3: Kepler-poinsot solids

Cylinders, Spheres, and Cones

Definition: Cylinders



Cylinders are prisms with two circular faces. Their volume is given by

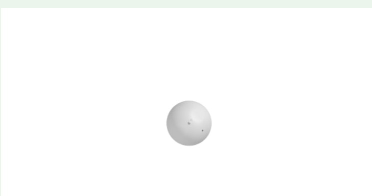
$$V_{cylinder} = \pi r^2 h \tag{4.3.1}$$

and surface area by

$$S_{cylinder} = (2\pi r)(r+h) \tag{4.3.2}$$

where r is the radius of the circular faces, and h is the distance between the two circular faces of the cylinder, or its height. A cylinder does not need perfect circles as its bases, provided both are congruent. The most common alternative to a circular base is an elliptical one. A shape with this base would be called an elliptical cylinder.

Definition: Spheres



Spheres are perfectly round objects that are found in a 3-dimensional Euclidean space. Technically, they have no thickness and are hollow. The volume described by a sphere is given by

$$V_{sphere} = \frac{4}{3}\pi r^3 \tag{4.3.3}$$

here r is the radius of the sphere. The surface area of a sphere is given by

$$S_{sphere} = 4\pi r^2. \tag{4.3.4}$$

Definition: Cones

Cones are pyramids with circular bases. Their volume is given by

$$V_{cone} = \frac{1}{3} A_B h \tag{4.3.5}$$

where h is the height of the cone and A_B is the area of the base. The surface area of a cone is described by the formula $S = \pi r^2 + \pi r \sqrt{}(r^2 + h^2)$

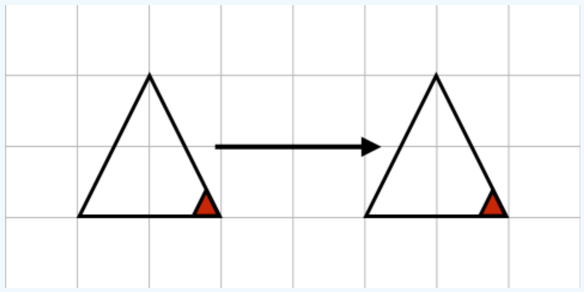
4.4: Transformations

Rigid Motions

Translations

A **translation** is a transformation that moves a figure (without altering dimensions) to a new position.

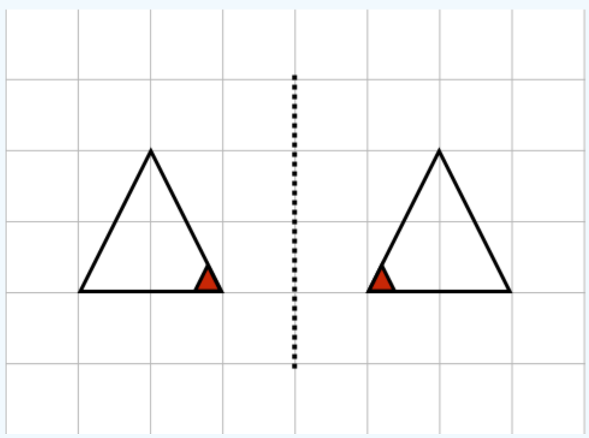
Example 4.4.1:



Reflections

A **reflection** is a transformation that maps a figure so that a line, called the **line of reflection,** is the perpendicular bisector of every line segment joining a point on the figure and the corresponding point on the reflected figure

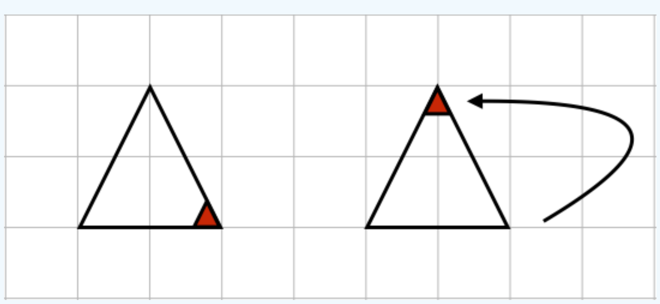
Example 4.4.2:



Rotations

Rotations are transformations where a figure is rotated about its center by a specified amount (given usually in degrees).

Example 4.4.3:



Thinking Out Loud:

Are reflections, translations, and rotations related?

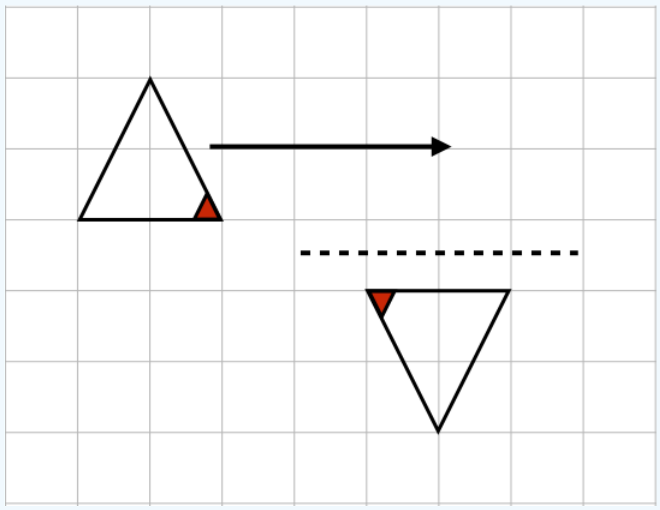
How are the operations of translation, reflection, and rotation like the arithmetic operations?

Composite transformations

Glide reflections

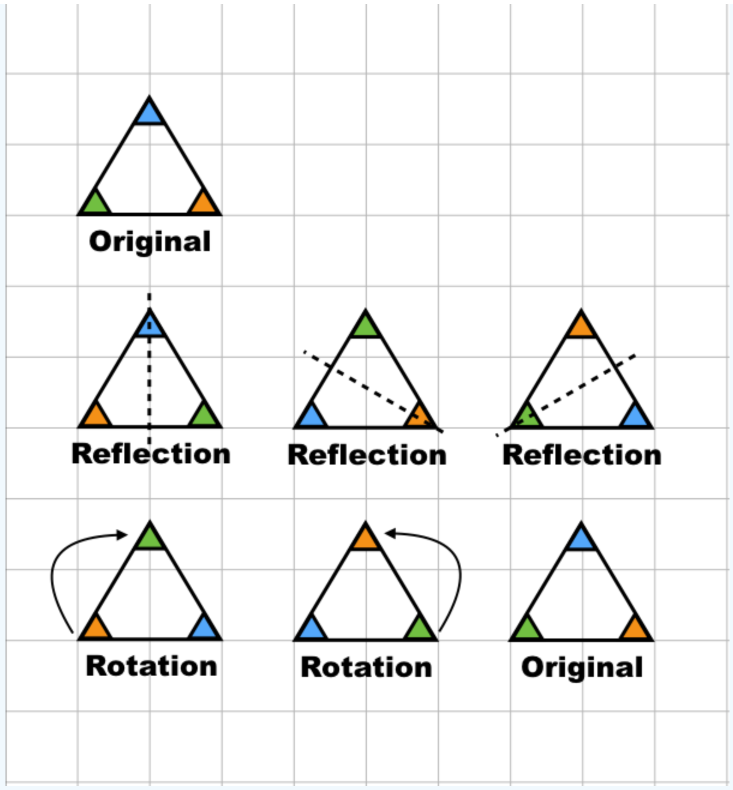
Glide reflections are a translation followed by a reflection with the condition that the translation vector and the line of reflection are parallel (that is, point in the same direction).

Example $4.4.4$:



Example $4.4.5$:

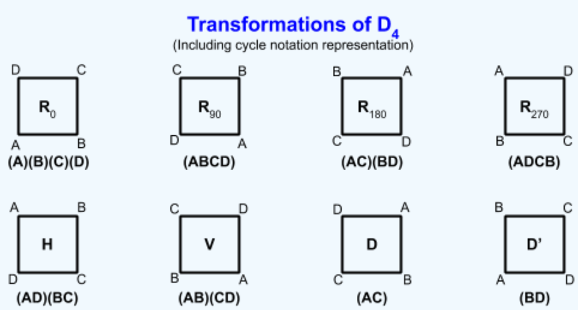
Consider the equilateral triangle below. What rigid motions can we do to it that will result in the triangle occupying the same space? What happens if we combine rigid motions? Keep in mind that "doing nothing" is also a rigid motion.



Exercise 4.4.6

Consider a square. What rigid motions can we make to it that will result in the square occupying the same space? What happens if we combine rigid motions? Keep in mind that "doing nothing" is also a rigid motion.

Answer



4.5: Symmetry

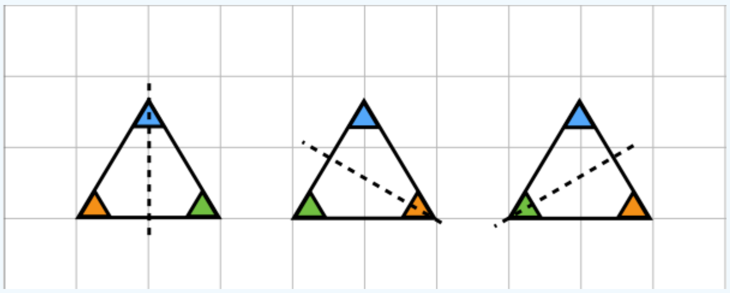
In 2D geometry, a figure is symmetrical if an operation can be done to it that leaves the figure occupying an identical physical space. This can be accomplished in two ways.

Line symmetry

Line symmetry occurs when a line may be passed through an object such that both halves of the object perfectly mirror each other.

Example 4.5.1:

Consider the triangle below. An equilateral triangle has three instances of line symmetry: one from each vertex to the midpoint of the opposite side.

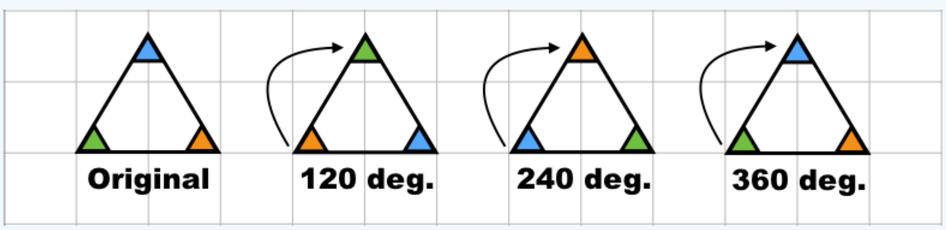


Rotational Symmetry

Rotational symmetry occurs when a shape may be rotated to occupy the same space as the original.

Example 4.5.2:

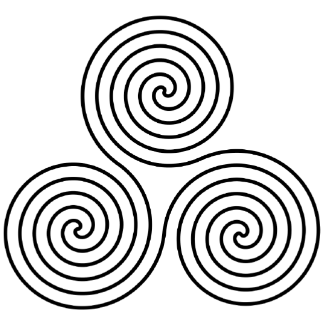
Let's take the same triangle again. An equilateral triangle has three degrees of rotational symmetry: at 120, 240, and 360 degrees. Once we have determined rotational symmetry to 360 degrees, we can stop, as the pattern will repeat itself after that.



Symmetry in Art

Symmetry in art is used as a way to emphasize beauty and order. The human brain finds symmetry to be attractive and beautiful. For this reason, artists attempt to emphasize (at certain times and using certain methods) the symmetry of their subject or surroundings.

Triskelions - an example of rotational symmetry.



4.6: Summary

Algebra

The algebra of a given object has rules and procedures specific to that object - the operations and rules of sets aren't exactly the same as the ones of numbers or statements. Taking mathematics in primary and secondary education, we are familiar with the algebra of numbers. Now we can apply those same thought processes to other areas of mathematics to prove statements or identities, and determine solutions for abstract problems.

Example 4.6.1:

An "object" might be said to be "all numbers", or "the Universal set containing all numbers." In algebra, we know that we can do operations to numbers: addition, subtraction, multiplication, and division, among others.

These operations are closed to numbers: when we do addition with numbers, we will always receive a number as the result.

This is true of other objects also: statements, sets, and geometric shapes are all objects with their own distinct operations and properties.

In mathematics, we view algebra as the study of relationships between objects, as well as the study of groups of objects. We are already familiar with some relationships between specific quantities.

Example 4.6.2:

In physics:

- $F = ma$, where F is force, in newtons, m is mass, in kg, and a is acceleration
- $v = \frac{d}{t}$

We also know that these formulas can be manipulated algebraically - that is, that the formula defines the relationship between the elements of the formula, and can be used to determine any of the elements as required.

Some Objects, Operations, and Properties

Objects	Numbers	Statements	Sets	Geometric shapes
Operations	$+,-,\cdot,\div$	$\neg,\vee,\wedge,\rightarrow$	$\subset,\subseteq,^c,\cup,\cap$	Reflection, rotation, translation
Properties	Closed Distributive	Closed Distributive	Closed Distributive	Closed Distributive

Closed operations are those that generate an answer in the same group as the elements operated upon.

Distributive operations are those that, in a mathematical sentence, yield the same result when FOILed into a bracket as when not.

Example 4.6.3:

$\neg(A \vee B) = \neg A \wedge \neg B$

$2(3+6) = 2(3) + 2(6)$

The Algebra of Geometry

In geometry, our operations are specific to the type of geometry we are doing. This means that the operations we use in 2-dimensional Euclidean geometry might not necessarily be the same as those we apply in 3-dimensional Euclidean or non-Euclidean geometry. For us, we will consider the 2-dimensional transformations as operations. Translations, reflections, and rotations follow the same rules as other operations: they are closed (when you move a shape, the result is a shape) as well as distributive (the order in which you move, rotate, or reflect a shape has no effect on the result).

4.E: Basic Concepts of Euclidean Geometry (Exercises)

Exercise $4.E.1$: Similarity

Assess whether each of the following statement is true or false and justify your answer.

1. Any two equilateral triangles are similar.
2. Any two isosceles triangles are similar.
3. Any two squares are similar.
4. Any two rectangles are similar.
5. Any two pentagons are similar.
6. Any two polygons are similar.

Exercise $4.E.2$: Area

Starting with the formula for the area of a triangle, show how to obtain the formula for the area of a parallelogram and the area of a hexagon.

Exercise $4.E.3$: Interior angle

Find the general formula for the center angle in a regular n-sided polygon? Justify your answer.

Find the general formula for the exterior angle in a regular n-sided polygon? Justify your answer.

Exercise $4.E.4$: Triangles

Prove or disprove the following: an equilateral triangle is an isosceles triangle.

explain why an equilateral triangle is not a scalene triangle?

Exercise $4.E.5$: True or False

Explain why the following statements below are true.

1. A square is a rectangle.
2. A rectangle is a parallelogram.
3. A square is a kite.
4. A parallelogram is a trapezoid.

Exercise $4.E.6$: Angles

Given cutout sheets with angles $40^\circ, 55^\circ$ and 85°. By adding or subtracting angles, construct other angles that measure $15^\circ, 30^\circ, 70^\circ, 95^\circ$and 100°.

Can you construct an angle that measures 75°?Explain how or say why not.

Exercise $4.E.7$: Converse of the Pythagorean Theorem

State the statement for the converse of the Pythagorean Theorem. Is this statement true or false? Justify your answer.

Exercise $4.E.8$: Venn diagram

Create a Venn diagram to illustrate the types of quadrilaterals, listed in this section.

Exercise $4.E.9$: Parallelogram

Let $(a;b);((0;c);(d:e);(f;0)$ be vertices of a quadrilateral. Show that if you take the midpoints of any quadrilateral and connect them in turn, you will always get a parallelogram.

Exercise $4.E.10$: Venn Triangles

Express the relationship between scalene, isosceles and equilateral triangles with Venn diagram.

Exercise $4.E.11$: Angles

Consider the figure:

Given that \(a= 47^{\circ}, \) and \(c= 55^{circ}.\)

Find other listed angles.

Exercise $4.E.12$:

Refer to the cube picture above. By Illuyanka (Own work) [GFDL (http://www.gnu.org/copyleft/fdl.html)], via Wikimedia Commons

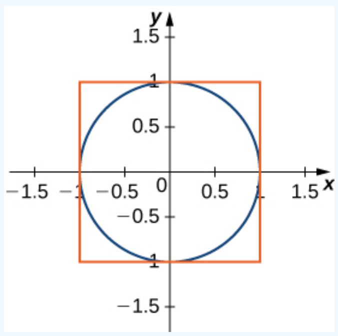
Name the following:

1.Two parallel line segments.

2.Two line segments that do not lie in the same plane. 3.Two intersecting line segments. 4.Three concurrent line segments that do not lie in the same plane. 5.Two skew line segments. 6.A pair of supplementary angles. 7.A pair of perpendicular line segments.

Exercise $4.E.13$: Inner Circle

Find the area between the perimeter of this square and the unit circle.

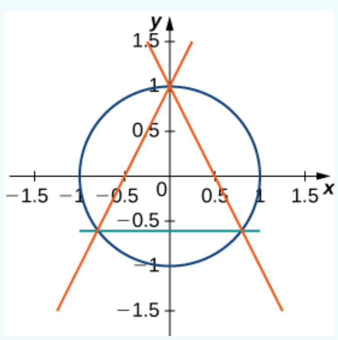


Answer

$1-\pi$

Exercise $4.E.14$: Outer Circle

Find the area between the perimeter of the unit circle and the triangle created from connecting the points $(0,1)$, $(-\frac{4}{5},-\frac{3}{5})$and $(\frac{4}{5},-\frac{3}{5})$, as seen in the following figure.



Answer

(\displaystyle π−\frac{32}{25}\)

Exercise $4.E.15$: Reuleaux triangle

The Reuleaux triangle consists of an equilateral triangle and three regions, each of them bounded by a side of the triangle and an arc of a circle of radius \(s \) centered at the opposite vertex of the triangle. Show that the area of the Reuleaux triangle in the following figure of side length s is $\frac{s^2}{2}(\pi-\sqrt{3})$.

An equilateral triangle with additional regions consisting of three arcs of a circle with radius equal to the length of the side of the triangle. These arcs connect two adjacent vertices, and the radius is taken from the opposite vertex.

Exercise $4.E.16$:lunes of Alhazen

Show that the area of the lunes of Alhazen, the two blue lunes in the following figure, is the same as the area of the right triangle *ABC*. The outer boundaries of the lunes are semicircles of diameters AB and AC respectively, and the inner boundaries are formed by the circumcircle of the triangle ABC.

A right triangle with points A, B, and C. Point B has the right angle. There are two lunes drawn from A to B and from B to C with outer diameters AB and AC, respectively, and with the inner boundaries formed by the circumcircle of the triangle ABC.

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Exercises 13-16 are from

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CHAPTER OVERVIEW

5: Basic Concepts of Probability

Learning Objectives

Develop the students:

- ability to do the counting
- probabilistic reasoning

Thinking out Loud

What do we mean if we say that the probability of tossing a coin and seeing a head is about 0.5 (50% or the odds are 50/50)?

One possible answer: We are claiming that if you toss the coin in question a large number of times then you should see heads appearing about half of the time.

Thinking out loud

What do people mean when they refer to a "100-year" flood?

One possible answer: They are claiming that the likelihood of a flood of that magnitude happening in any given year is 0.01 (1%, or the odds are 1/100).

5.1: Counting
5.2: Probability: Living with odds
5.3: Expected value
5.E: Basic Concepts of Probability (Exercises)

Contributor

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5.1: Counting

Most of us think that counting is as easy as 1, 2, 3... When counting objects, one needs to be careful to not count an object more than once or miss an object. In this section, we will explore some ideas behind counting.

The Multiplication Principle

If a process can be broken down into two steps, performed in order, with m ways of completing the first step and n ways of completing the second step after the first step is completed, then there are (m)(n) ways of completing the process.

Example 5.1.1:

Suppose that pizza can be ordered in 3 sizes, 2 crust choices, 4 choices of toppings, and 2 choices of cheese toppings. How many different ways can a pizza be ordered?

Solution

To determine the number of possibilities, we will use the multiplication principle. Let $S =$ pizza size, $C =$ crust choice, $T =$ topping choice, and $Ch =$ cheese choice.

Since we need to choose one choice from each category, we can write that we need to choose size, and crust, and topping, and cheese.

Let's use the multiplication principle:

$$\begin{aligned} \text{Ways} &= (S)(C)(T)(Ch) \\ &= (3)(2)(4)(2) \\ &= 48 \end{aligned}$$

What if there was only one choice for cheese? How would this affect the calculation?

Example 5.1.2:

Count the number of possible outcomes when:

1. A coin tossed four times.
2. A standard die is rolled five times.

Permutation

Definition: Permutation

A permutation is an ordered arrangement of objects.

The number of permutations of n distinct objects, taken all together, is $n!$, where

$$n! = n(n-1)(n-2)\ldots..1 \tag{5.1.1}$$

.

Note that $0! = 1$.

Example 5.1.3

Miss James wants to seat 30 of her students in a row for a class picture. How many different seating arrangements are there? 17 of Miss James' students are girls and 13 are boys. In how many different ways can she seat 17 girls together on the left, then the 13 boys together on the right?

Solution

Let's start with the girls. There are 17 of them, and so, when seating the first girl in the row, there are 17 choices. The next spot will have 16 choices left, then 15, and so on. Thus, the number of choices for seating the girls can be written $17!$.

For the boys, by the same reasoning, there are $13!$ ways to seat them on the right.

Now let's apply the multiplication principle: we need to seat the girls and the boys at the same time. For each permutation we might pick for the girls, we need to apply each different case for the boys as a distinct possibility. So, our result is $(17!)(13!)$. This means there are $2.215 \cdot 10^{24}$ different ways to seat these students with girls on the left and boys on the right!

Permutations

The number of permutations of r objects picked from n objects, where $0 \leq r \leq n$, is

$$ {}_nP_r = \frac{n!}{(n-r)!}. \tag{5.1.2}$$

When reading this out loud, we say "n Pick r" - when we pick something, like a team for sports or favorite desserts, the order matters.

Example 5.1.4:

Using the digits 1,3,5,7, and 9, with no repetitions of digits, how many three–digit numbers can be made?

Solution

We have $n = 5$ objects, and we want to pick $r = 3$ of them. So via Equation 5.1.2:

$$\begin{aligned} {}_nP_r &= \frac{5!}{(5-3)!} \\ &= \frac{5!}{2!} \\ &= \frac{(5)(4)(3)(2)(1)}{(2)(1)} \\ &= (5)(4)(3) \\ &= 60 \end{aligned}$$

Combination

The following is defined already in 3.3 Finite Difference Calculus.

Combinations

The number of combinations of r objects chosen from n objects, where $0 \leq r \leq n$, is

$$ {}_nC_r = \frac{n!}{(n-r)!r!} \tag{5.1.3}$$

${}_nC_r$ is also denoted as $\binom{n}{r}$. When reading this out loud, we say "n Choose r" - when we choose objects, like candies out of a bag or clothes from a closet, the order doesn't matter.

Example 5.1.5:

Evaluate ${}_6C_2$, and ${}_4C_4$.

Solution

Let's try ${}_6C_2$, or $\binom{6}{2}$:

$$ {}_nC_r = \frac{6!}{(6-2)!2!}$$

$$_nC_r = \frac{6!}{(4)!2!}$$

$$_nC_r = \frac{(6)(5)(4)(3)(2)}{(4)(3)(2)(2)}$$

$$_nC_r = \frac{(6)(5)}{(2)}$$

$$_nC_r = \frac{30}{2}$$

$$_nC_r = 15$$

Now let's tackle $\binom{4}{4}$:

$$_nC_r = \frac{4!}{(4-4)!4!}$$

$$_nC_r = \frac{4!}{0!4!}$$

The result of $0!$ is 1.

$$_nC_r = \frac{4!}{4!}$$

$$_nC_r = 1$$

This makes sense: there is only one way to choose four things from a group of four things. You choose all of them, and that is the only option.

Example 5.1.6:

How many 5-member committees are possible if we are choosing members from a group of 30 people?

Let's see: we have 30 people to choose from, so $n = 30$. We want to choose 5 members, so $r = 5$. Lastly, we don't care about the order in which we choose, so we use $_nC_r$:

$$\binom{30}{5} = \frac{30!}{(30-5)!5!}$$

$$\binom{30}{5} = \frac{30!}{25!5!}$$

$$\binom{30}{5} = \frac{(30)(29)(28)(27)(26)}{5!}$$

$$\binom{30}{5} = \frac{(30)(29)(28)(27)(26)}{120}$$

$$\binom{30}{5} = 142506$$

Example 5.1.7:

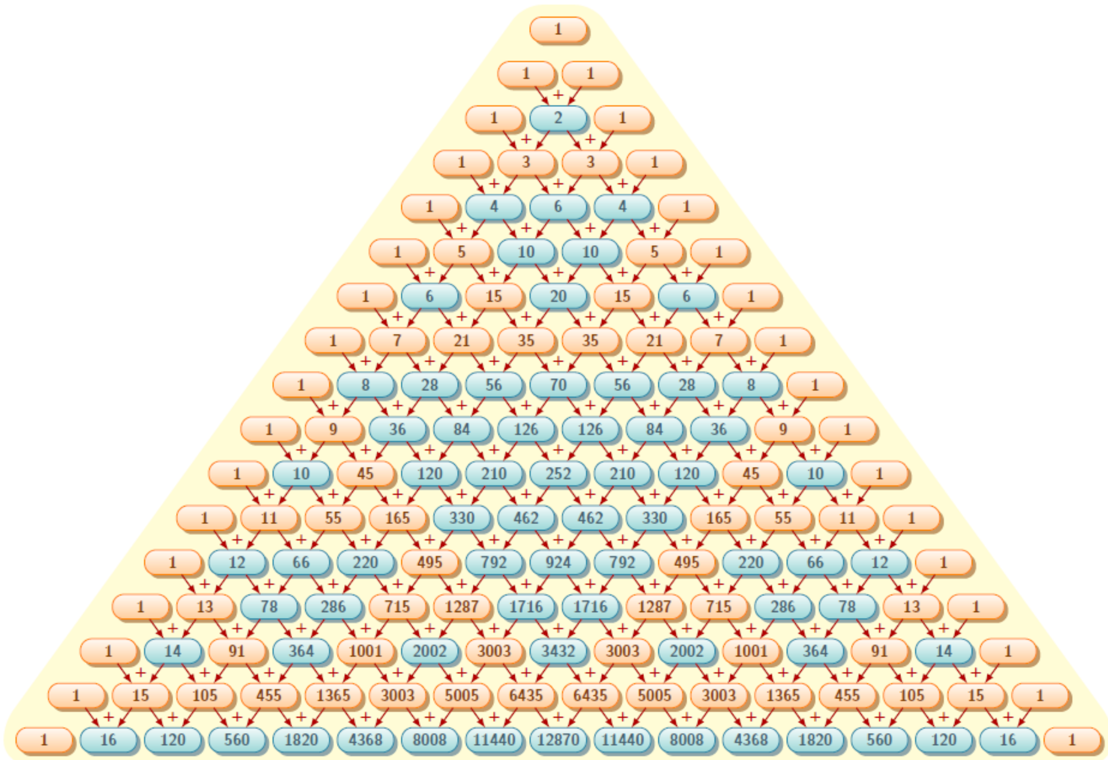
1. In how many ways can 3 men and 3 women sit in a row, if no two men and no two women are next to each other?
2. In how many ways can 3 men and 3 women sit in a circle, if no two men and no two women are next to each other?

Pascal's Triangle

Pascal's triangle was developed by the mathematician Blaise Pascal. It is generated by adding the two terms diagonally above to receive the new term, where the first term is 1, which is defined as $_nC_r = _{n-1}C_{r-1} + _{n-1}C_r$.

The triangle is useful when calculating $_nC_r$ as well: count down n rows, and then count in r terms. For example: $_7C_2$ means that we look at row 7, term 2: 6.

- Gives the coefficients of $(a+b)^n$.
- The entries of the nth row are $C(n,0), C(n,1)...C(n,n)$
- The sums of each row are consecutive powers of 2.
- The third element from each row yields triangular numbers.



Binomial Expansion

$(x+y)^n = x^n + nx^{(n-1)}y + \cdots + \binom{n}{k} x^k y^{(n-k)} + \cdots + y^n$.

5.2: Probability: Living with odds

Probability is a subtle concept: There are several different things we mean by probable.

Randomness is also subtle. Ten heads in a row? Did we cheat? We probably did, but don't know yet as of when we typed this. In order to get ten heads in a row, we expect to have to cheat, but we don't know for sure that we will have to.

Our knowledge of things to come is also imperfect. What can we say in the face of imperfect knowledge? How can we reason knowing our knowledge is imperfect?

Definitions:

Outcome: An outcome (simple outcome) is the most basic possible result of observations or experiments.

Sample Space: Set of all possible outcomes.

Events: An event is one or more outcomes that share a property of interest

Example 5.2.1:

In a coin toss, the possible outcomes are T, H.

In tossing two coins the outcomes are TT, TH, HT, HH.

In rolling a standard die the outcomes are 1, 2, 3, 4, 5 or 6 dots.

In an experiment on colours of bred pea plants, the outcomes could be a red, white or pink plant.

Types of Probability:

Experimental: We observe over a length of time or perform an experiment many times and calculate the relative frequency of the event. The relative frequency is the number of times the desired outcome occurs per the number of times an experiment is performed or observations made – it's a percentage!

Theoretical: Based on a model where all outcomes are equally likely.

Subjective: Estimate based on intuition or experience (ideally to be made by an expert in the field who also has a sound grasp of probability).

Fundamentals of Probability

Example 5.2.2:

- When tossing two coins we might be interested in the event of two heads, or the event of at least one head.
- When rolling a standard die we might be interested in the event of rolling a one or the event of rolling an even number or the event of rolling a number less than 4.
- When breeding 6 pea plants we might be interested in the event all plants are the same colour.

Example 5.2.3:

Experiment: Toss two coins

Outcomes: TT, TH, HT, HH

Event: getting one head consists of TH or HT

Simple Calculations

Assumptions: a fair coin, a fair dice, well-shuffled cards

Method: Count the total number of possible outcomes n and count the number of outcomes in the event A.

$P(A) = \frac{|A|}{n}$.

In general, the probability of an event, A, occurring is $P(A) = \frac{|A|}{|S|}$, where $|S|$ is the total number of outcomes in the sample space S.

Formal Properties of Probability

Rules:

Let A, B, C, … be events. P(A) denotes the probability event A occurs.

RULE #1: $0 \leq P(A) \leq 1$ .for every event A. The probability of any event is a number between 0 and 1.

Rule # 2: P(E) = 0 if and only if the event is impossible.

Rule# 3: P(E) = 1 if and only if the event is a certainty.

Rule #4: $P(E^c) = 1 - P(E)$. (The probability of an event NOT occurring is 1 – probability the event occurs).

Thinking Out Loud:

Probability can also be expressed as a percentage. What is the range?

Example 5.2.4:

What is the probability of rolling a four on two six-sided dice?

There are 36 different ways of rolling two six-sided dice. (How do we know?)

There are 3 ways of rolling four (1+3, 2+2, 3+1)

So, the probability is 3/36 = 1/12

Thinking Out Loud:

What is the probability of flipping 10 coins and getting all the same?

What is the probability of flipping 10 coins and getting all heads?

Example 5.2.5:

From a well-shuffled deck of 52 cards, three cards are drawn at random. Find the probability that **exactly** one King will be drawn.

There are $\binom{52}{3}$ ways to select 3 cards out of 52. There are $\binom{4}{1}$ ways to select one King out of four Kings. Further, there are $\binom{48}{2}$ ways to select the remaining two non-King cards out of 48 non-King cards. Therefore, the probability that **exactly** one King will be drawn is

$$\frac{\binom{4}{1} \cdot \binom{48}{2}}{\binom{52}{3}} \tag{5.2.1}$$

.

Example 5.2.6:

From a well-shuffled deck of 52 cards, three cards are drawn at random. Find the probability that **at least** one King will be drawn.

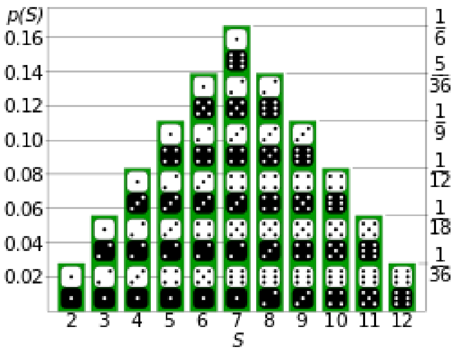
The probability is the sum of the probability that **exactly** one, two, and three King will be drawn. Hence ,

$$\frac{\binom{4}{1}\cdot\binom{48}{2}}{\binom{52}{3}}+\frac{\binom{4}{2}\cdot\binom{48}{1}}{\binom{52}{3}}+\frac{\binom{4}{3}\cdot\binom{48}{0}}{\binom{52}{3}} \tag{5.2.2}$$

.

Probability distribution

The probability distribution is a display of the probability for every possible event. For example, see the probability distribution of rolling two six-sided dice.



Combining Probabilities

Definition: Independent

Two events are independent if the outcome of one event does not affect the probability of the second occurring otherwise we say the two events are dependent

Example 5.2.7:

Tossing a coin twice, the event of the first toss being a head and the event of the second toss being a head are independent.

Tossing a coin twice, the event of the first toss being a head and the event of at least one head occurring are NOT independent.

Rules Continued:

RULE #5: If A and B are independent events then P(A and B) = P(A)P(B).

This rule generalizes to 3 or more independent events.:

Example 5.2.8:

Suppose you toss three coins. What is the probability of getting three tails? P(3 tails)=P(1tail) P(1tail) P(1tail)= 1/8.

Rules Continued:

RULE #6: If A and B are dependent events then P(A and B) = P(A)P(B given A)

{P(B given A) is the conditional probability of even B occurring when event A is assumed to have already occurred).}

Example 5.2.9:

Two members are selected from a pool of 17 male students and 23 female students. Find the probability that the first student selected is a male and the second is also male.

$P(A) =$ The first student selected is male.

$P(B) =$ The second student selected is male

$P(A) = \frac{17}{40}$

$P(B\, given\, A) = \frac{16}{39}$

$P\, A\, and\, B) = \left(\frac{17}{40}\right)\left(\frac{16}{39}\right)$

Definition: Mutually Exclusive

Two events are mutually exclusive (non-overlapping) if the two events can't occur at the same time. Two events are overlapping if they can occur at the same time.

Example 5.2.10:

Tossing a coin twice, the event of the first toss being a head and the event of the second toss being a head are overlapping. The outcome of HH would satisfy both events. Tossing a coin twice, the event of the first toss being a head and the event of no heads occurring are non-overlapping, since we can't have both occurring. When tossing a die, the event of tossing a number greater than 4 and the event of tossing an even number are overlapping events. An outcome of 6 would satisfy both.

Rules Continued:

Rule #7: If A and B are non-overlapping events then

P(A or B) = P(A)+ P(B)

This generalizes to 3 or more events.

Example 5.2.11:

Find the probability of rolling either a 1 or a 2 on a single die.

$P(1) = \frac{1}{6}$

$P(2) = \frac{1}{6}$

$P(1\, or\, 2) = \frac{1}{6} + \frac{1}{6}$

$P(1\, or\, 2) = \frac{2}{6} = \frac{1}{3}$

Rules Continued:

Rule #8: If A and B are overlapping events then

P(A or B) = P(A)+ P(B) - P(A and B).

Example 5.2.12:

Find the probability of drawing a queen or a heart from a standard deck.

$P(Q) = \frac{4}{52}$

$P(H) = \frac{13}{52}$

$P(Q\, and\, H) = \frac{1}{52}$

$P(Q\, or\, H) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52}$

$$P(Q\ or\ H) = \frac{16}{52} = \frac{4}{13}$$

Rules Continued:

Rule #9: Suppose the probability of an event in one trial is P(A). If all trials are independent, the probability that event occurs at least once in n trials is

$1 - P\left(no\,event\ A\ in\ n\ trials\right) = 1 - P(not\ A\ in\ one\ trial)^{n}$

Example 5.2.13:

Find the probability of getting at least one 6 in five rolls of a single die.

More on conditional probability

We have seen that P(A and B) = P(A)P(B given A)

I.e. P(B given A) = P(A and B)/P(A).

P(B given A) = the (conditional) probability that event B occurs given that event A is known to have occurred.

Example 5.2.14:

if a coin was tossed twice, P(HH) =$\frac{1}{2}$.

P(HH given both coins showed same value) = $\frac{1}{4}$. (the condition tells us it was either HH or TT).

Rules Continued:

Rule #10: The odds for event A occurring are:

Odds for event $A = P(A) : P(not\,A)$. This is usually converted to a pair of integers.

Example 5.2.15:

If you toss a coin twice, what are the odds you get at least one head appearing?

1:1 (50:50)

If the odds are 1:3 that a patient will survive heart surgery is that good or bad? What is it as a probability?

¼ = 0.25 = 25% not great

Example 5.2.16:

A fair coin is tossed 10 times. Which is more likely:

a. 10 heads in a row
b. THHTHTHTTH (giving 5 heads and 5 tails in total).

Both events are equally as likely- in each case, I explicitly predicted each toss of the coin! We confuse the fact that it is more likely to get 5 heads than 10 heads with a very specific outcome.

Coincidences

Partly because we don’t know probabilities as well as we should and partly because we over publicize uncommon events and attach significance to things after they happen we don’t appreciate coincidences properly.

When is your birthday: month and day?

It can be shown that in a room with 23 people there is a 50% chance two will share a birthday. In a room with 41 or more people, the probability is over 90%. (Note: if we specified in advance which day the birthday was on it would be far less likely to find two people with a birthday on that day).

Monty Hall Problem

Monty Hall runs a game show. Contestants pick one of three closed doors and win the prize behind the door. Behind one of the doors is a car, behind the other two doors, are pigs. After the contestant picks a door, Monty Hall opens one of the other doors to reveal one of the pigs. He asks the contestant if they would like to change their mind and take the other unopened door. Should they bother switching? Why?

5.3: Expected value

Definition:

For a probability distribution defined by $P(X=x)$, we define the expectation of the random variable X as

$$E(X)=\sum_{i=1}^{i=n} x_i P(X=x_i)$$
$$=x_1 P(X=x_2)+x_2 P(X=x_2)+\cdots+x_n P(X=x_n)$$

where x_i represents the observed outcome and $P(X=x_i)$ is the probability of the outcome occurring.

The “expected value of X” can be interpreted as the mean value of X.

The expectation values can be considered in two ways.

1. Long-run average

This is the measure one would see if the experiment was repeated a large number of times, namely $E(X)=np$, where n is the number of times the experiment occurred and p is the probability for the event to occur.

Example 5.3.1

If we tossed a coin 1500 times, and the random variable X, represents the number of heads observed, we would expect 750 heads, that is $E(X)=750$.

2. Probability weighted average

This is the measure that takes into account the relative probabilities of each observed outcome.

Example 5.3.2

For the probability distribution with random variable X defined by

x	2	3	4	5
$P(X=x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$E(X)=\sum_{i=1}^{i=4} x_i P(X=x_i)=(2\frac{1}{6})+(3\frac{1}{6})+(4\frac{1}{6})+(5\frac{1}{6})=\frac{15}{6}$.

Thus X has a mean value of $\frac{15}{6}$.

Example 5.3.3:

We toss 4 coins at the same time, then the probability of getting X number of tails:

x	0	1	2	3	4	Total
$P(X=x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$	

Then the expected value is

$E(X)=\sum_{i=0}^{i=4} x_i P(X=x_i)=(0\frac{1}{16})+(1\frac{4}{16})+(2\frac{6}{16})+(3\frac{4}{16})+(4\frac{1}{16})=\frac{32}{16}=2$.

Therefore, in the long run, we would expect to get 2 tails, when we toss 94 coins at the same time.

5.E: Basic Concepts of Probability (Exercises)

Exercise $5.E.1$: Counting

A typical PIN is a sequence of any seven symbols chosen from the 26 letters in the alphabets and the ten digits $0, 1, \ldots., 9$ with repetition allowed.

1. How many PIN's are available?
2. Supposing that symbols cannot be repeated, then how many PIN's are available?

Exercise $5.E.2$: Counting

How many (Canadian) postal codes would there be possible without repetition of letters or numbers?

Exercise $5.E.3$: Counting

Seven women and nine men are on the faculty in the Mathematics Department.

1. How many different committees are there which are made up of five members of the department, at least one of which is a woman?
2. How many ways can the five-person committee be arranged around a circular table?

Exercise $5.E.4$: Probability

You have 23 people in a room. What is the probability that the two of them have the same birthday?

Exercise $5.E.5$:Probability

A coin is tossed 7 times, find

1. the probability that we see at least 2 heads?
2. the probability that we see exactly 2 heads?
3. the probability that we see exactly 2 heads or exactly 4 heads?

Exercise $5.E.6$:Probability

What is the conditional probability that the sum of the dice is 10, 11, or 12 given that the first die rolled comes up a 6?

Exercise $5.E.7$:Probability

We have a standard deck of 52 cards and you select a card at random.

1. What is the probability you select a 5?
2. What is the probability you don't select a 5?
3. What is the probability you select a 5 or 7? Are these mutually exclusive events or not?
4. What is the probability you select a 5 or a diamond? Are these mutually exclusive events or not?
5. What is the probability you select the Jack of Spades given you have selected a face card?
6. What is the probability you select a diamond given you have selected a red card?
7. What is the probability you selected a red card given you've selected a diamond?

Now suppose you select two cards from the deck.

8. What is the probability you select two Jacks assuming you replace a card and reshuffle the deck before selecting again? Are these dependent or independent events?
9. What is the probability you select two Jacks assuming you don't replace a card once you've selected it? Are these dependent or independent events?
10. What is the probability you select a Jack and an Ace – again assuming replacement. Careful- you could select the Jack then the Ace or the Ace then the Jack.

Exercise $5.E.8$: Probability

In a town of 10,000 people, 400 have beards (all men), 4000 are adult men, and 5 of the townspeople are murderers. All 5 murderers are men and 4 of the murderers have beards.

Suppose you go to this town and select a towns person at random.

- Let A be the event that the person turns out to be one of the five murderers.
- Let B be the event the person is bearded.
- Let C be the event the person is an adult male.

Find P(A), P(A given B), P(B given A), P(A given not B), P(A given C), P(A given not C).

Exercise $5.E.9$: False Positive

Disease X is a disease affecting about 1 percent of the population. A test for Disease X will test positive on all afflicted with the disease and will also test positive for 5% of the population who do not have the disease.

1. What is the probability a randomly chosen person has disease X?
2. What is the probability a randomly chosen person will test positive for the disease?
3. Suppose you test positive for the disease. What is the probability you don't actually have the disease? (This is the conditional probability that you don't have the disease given you tested positive for it).
4. What would be the probability that you test positive for the disease twice given that you don't have disease X?
5. Can you identify (at least 2) criticism's of our theoretical probability calculations here?

Exercise $5.E.10$: Probability

1. the probability that we see at least 2 heads?
2. the probability that we see exactly 2 heads?
3. the probability that we see exactly 2 heads or exactly 4 heads?

Exercise $5.E.11$: Probability

A multiple-choice test has 10 questions, each with 4 possible answers. A student guesses all ten questions.

a. Find the probability that the student will get all ten questions right.
b. Find the probability that the student will get **at least 1** question right.

Exercise $5.E.12$: Probability

Suppose two fair dice are thrown.

1. What is the probability the sum of the dice is at most 4?
2. What is the probability the sum of the dice is more than 4?
3. What are the odds the sum of the dice is at most 4?
4. What is the probability the sum of the dice is at most 4 given that the first die shows a 3?

Exercise $5.E.13$: Combination

If $n \geq k+2$ and $k \geq 2$, show that $\binom{n}{k} - \binom{n-2}{k} - \binom{n-2}{k-2}$ is even.

Exercise $5.E.14$: Combination

Find the coefficient of x^5 in the binomial expansion of

$$\left(\frac{2}{x} + x^2\right)^{25} \tag{5.E.1}$$

.

Exercise $5.E.15$: Combination

For natural numbers n and r, $r < n$, show that

$$\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}. \tag{5.E.2}$$

Answer

under construction.

Exercise $5.E.16$:

A fair coin, a double-headed coin and a double-tailed coin are placed in a bag. A coin is randomly selected. The coin selected is then tossed.

1. Find the probability that the coin lands with a "head".
2. When the coin is tossed, it lands "tail". Find the probability that it is the double-tailed coin.

Answer

$\frac{1}{2}\frac{3}{4}$.

CHAPTER OVERVIEW

6: Introduction to Statistics

Learning Objectives

Develop the students:

- ability to study data
- statistical reasoning

Statistics is the science of collecting, analyzing, and drawing conclusions from data. There are two branches of statistics.

Branches of Statistics

Definition: Descriptive Statistics

Descriptive statistics is the branch of statistics used in describing the data via graphs, tables or other statistical measures. This is the type of statistics we do when we have a lot of data and want to summarize it appropriately.

Definition: Inferential Statistics

Inferential statistics is the branch of statistics that deals with inferring/estimating population characteristics from sample data. If a sample represents a given population accurately, then analyzing the sample can lead to significant conclusions about the population as a whole.

Terminology

Definition:Population

The population is the entire group that a statistical sample is drawn. If an accurate sample is drawn, significant hypotheses can be developed with reference to the entire population.

Definition: Sample

A sample is a set of data collected from a specific population by a defined procedure.

Common Sampling Techniques

- Simple random sampling
- Systematic sampling
- Convenience sampling, which is poorly named
- Stratified sampling

Note that one could use a combination of these methods – for example, stopping every 5^{th} person to participate in a survey is systematic and convenience sampling and all of these methods depend on randomness in some way or another!

Bias

A study suffers from **bias** if its design or conduct tends to favour certain results. It can happen as a result of failing to choose a truly representative sample **(selection bias/participation bias)**. Bias might also be present if the person conducting the study is biased (by having a personal stake in the study, by having strong beliefs or expectations on the subject), even if their bias is subconscious. It can even happen at the end of the study if the data is intentionally or unintentionally distorted to lead to a particular conclusion. Finally, there could be a flaw in the study conduct, resulting from a systematic measuring error for example.

Placebos

In an experiment, a **placebo** is a 'phony treatment' that is often given to the control group. On the surface (to a patient and possibly data collector) it appears identical to the treatment under study but it is missing the 'active ingredient' understudy. The **placebo effect** refers to the situation when patients improve just because they believe they are receiving a useful treatment.

Confounding Variables

Variables that are not intended to be part of a study that confound (confuse) a study's results are called confounding variables.

Surveys and Opinion Polls

They are a type of observational study with their own special issues one should be aware of. Margins of error, confidence intervals, and confidence levels are often reported with opinion polls and survey results.

Definition: margin of error, confidence interval, and confidence level

The **margin of error** is a number or percentage which should be added and subtracted from the reported number in order to provide a range of numbers in which the actual number probably resides. This range of numbers is called the **confidence interval**. Margins of error and confidence intervals are always calculated with respect to a particular **confidence level**. Usually, the confidence level used is 95% (19/20). This means that we can be 95% confident that the confidence interval contains the correct value. The margin of error for 95% confidence is approximately equal to $\frac{1}{\sqrt{(N)}}$, where N is the size of the sample. As N increases, the margin of error decreases

Topic hierarchy

6.1: Qualitative Data and Quantitative Data
6.2: Descriptive Statitics:Measures of Center, Measures of Variation and the Five -Number Summary
6.3: Introduction to Statistical Calculations using Microsoft EXCEL
6.4: Binomial distribution and Normal Distribution
6.E: Introduction to Statistics (Exercises)

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6.1: Qualitative Data and Quantitative Data

Data

There are two types of data that we can collect:

- Qualitative data describes a subject, and cannot be expressed as a number.
- Quantitative data defines a subject and is expressed as a number (it can be quantified) that can be analyzed. There are two types of quantitative data continuous and discrete.

Example 6.1.1:

1. Ratings of a tv show
2. Grades of an exam.
3. Marks of an exam.
4. Students heights in a class

Graphs

There are many types, including:

1. Pie charts and bar graphs are used for qualitative data
2. Histograms (similar to bar graphs) are used for quantitative data
3. Line graphs are used for quantitative data
4. Scatter graphs are used for quantitative data

Graphs should contain:

- A descriptive title below the graph or chart
- A caption below the title (optional)
- Axes labelled with the name of variable, units (if applicable) and the variable intervals; intervals must be spaced according to scale
- A legend to indicate which data points belong to which set of data, if more than one data set is displayed

Example 6.1.2:

Summarizing Data

There are various ways to summarize a data set:

- Distribution tables
- Graphs of raw data
- Sample statistics such as mean, median, mode, standard error and standard deviation
- Graphs based on average values with error bars to indicate a standard error or standard deviation

Simple Descriptive Statistics

Descriptive statistics are numbers and processes that describe a group of data. The most common descriptive statistics focus on determining the "average" of the data. However, there is more than one "average," so we must be specific when finding them. Values which describe the “average” are:

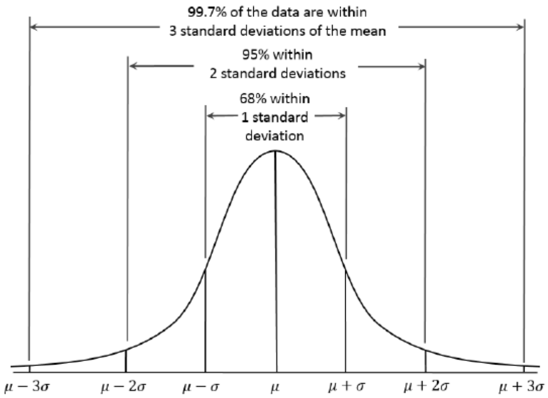
- Mean - the sum of all the data divided by the number of datum in the group, the "average" that most people mean (see what we did there?).
- Median - the middle-most datum, when all the data are arranged by the quantity
- Mode - the most common datum in the set

As you can imagine, data sets very rarely are all one value. Thus, we need to describe how the data is arranged around the mean. Values which describe the variation of the data around the mean are:

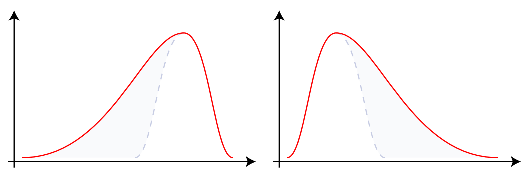
- Standard deviation describes, for a whole population, the dispersion of the whole population's data set.
- Standard error describes, for a representative sample's data set, the standard deviation of that set.

The Normal Distribution and Other Shapes of Data Distribution

Analysis of many phenomena results in a ***normal*** distribution of data. Normal distribution approximates a bell-shaped curve when data is plotted on a line graph. As the number of replicates in a data set increases, the graph approaches a perfect bell shape, so the mean, median and mode are all at the peak of the curve



A distribution may be skewed due to a disproportionate number of extremely high or low values, especially if the sample size is small.



A distribution may show more or less variation, around the mean. This is quantified by the size of the standard deviation of the distribution.

It is important to remember that not all types of data distributions have a single peak.

6.2: Descriptive Statitics:Measures of Center, Measures of Variation and the Five - Number Summary

Measures of Central Tendency

Definition: Mean, Median, Mode

- **Mean**: Add each data value and divide by the number of data values.
- **Median**: Arrange the data values in numerical order. The median is the middle data value. If there is an even number of data, then find the mean of the two closest to the middle.
- **Mode**: The data value that occurs most often.

Example 6.2.1:

Given the following set A, determine the mean, median, and mode of the set.

$A = 1, 1, 1, 2, 3, 5, 5, 7, 9, 12, 23$

$Mean = \frac{\Sigma A}{n}$, or the sum of all terms of A divided by the number of terms in A

$= \frac{1+1+1+2+3+5+5+7+9+12+23}{11}$

$= \frac{69}{11}$

$= 6\frac{3}{11}$

$Median = 5$

$Mode = 1$

Measurements of Variation

Measurements of variation are well named. These quantities describe how far apart the data points can be from each other. If a data set is imagined as a bull's eye, measurements of variation will describe the size of the target, as well as where there are groups of points or gaps in points. We use the following measures to describe the dispersion of data:

Definition: Measures of Dispersion

- **Range** describes the span of the data, or how far apart the biggest and smallest values are. It is calculated by subtracting the minimum value from the maximum value
- **Clusters** occur when groups of data occur together, and apart from the rest of the data points. There may be one or more clusters in any given data set.
- **Gaps** are places where data is expected to occur but does not.
- **Outliers** are data points which occur individually and do not behave according to the trend described by the rest of the data.
- **Standard Deviation**, given by s, describes how far, on average, observed data is from the expected mean.

Where n is the number of observations, we can determine a number of quantities:

Definition: Describing the Data

Variance describes how far from the average a set of values in a data set is expected to fall. Variance = s^2

The **first quartile** (**Q_1**) is the median of the part of the entire data set that lies at or below the median of the data set.

The **second quartile** (**Q_2**) is the median of the data set.

The **third quartile** (**Q_3**) is the median of the part of the entire data set that lies at or above the median of the data set.

Interquartile Range describes the difference between the first and third quartiles. IQR = $Q_3 – Q_1$

Five-Number Summary consists of five numbers that describe a data set:

1. The data's minimum value
2. The first quartile
3. The median
4. The third quartile
5. The data's maximum value

There are many ways in which a set of data can be distributed. In this course, we will focus on five distributions: uniform, skewed to the right, skewed to the left, bimodal, and normal.

Example 6.2.2:

Given data set B, give the five-number summary of the set.

$B = 1, 1, 2, 2, 4, 5, 5, 6, 6, 6, 6, 7, 8, 8, 9, 9, 9, 10, 12, 14, 16, 22, 29$

Remember, the five-number summary of a set is

1. The data's minimum value
2. The first quartile
3. The median
4. The third quartile
5. The data's maximum value

Let's start with the minimum, maximum, and median values, as those are the simplest.

1. The minimum value is 1.

3. The median value is the twelfth value (there are 23 values in all): 7

5. The maximum value is 29.

For the first and third quartiles, things get a little more complicated. When determining the first quartile, we include the median. When we switch to the third quartile, however, we cannot use the median again. Whether we use the median for the first or third quartiles is an arbitrary choice, but since Microsoft Excel uses the median to determine the first quartile, that's what we will do. Not all software does this, so you should be aware that things might not always be the same.

2. The first quartile is the average of the sixth and seventh terms (there are 12 terms in the first half, median included): $\frac{5+5}{2} = 5$

4. The third quartile is the 18th term (there are 11 terms in the second half, median excluded, and 12 plus six is 18): 10

So, the five-number summary is:

$1,\ 5,\ 7,\ 10,\ 29$

Exercise 6.2.1:Five number summary

Given $3, 2, 0, 5, 5, 3, 1, 0, 3, 2$

Obtain the five-number summary for these data.

a. Identify potential outliers, if any.
b. Construct a boxplot.

Note:

Wh**at is the meaning of the interpolated median? How does it differ from the median?**

6.3: Introduction to Statistical Calculations using Microsoft EXCEL

Example 6.3.1: In Class project.

We will collect the data in class.

Instructions:

1. Open a new Excel worksheet and save as Lab1.xls – I'd like the finished version submitted to the Blackboard site digital drop box. Put your name (first and last) in Cell A1. If you choose to work with a partner put the second name in Cell A2 and only submit one worksheet between you.

Use the appropriate decimals button to reduce all answers to at most 3 decimals. Save often!

Use Excel functions to calculate:

a. The mean
b. The median
c. The mode, Explain your mode answer!
d. The standard deviation
e. The range
f. The interquartile numbers
g. The Five number summary.

The functions you'll need are the average(cell range), median (cell range), mode (cell range), max(cell range), min(cell range), stdev(cell range), Quartile (cell range, quartile number). Try using the function button to access the functions instead of typing them out.

There is an easier way to get the above information and more!

2. Go to the file Button (top left hand corner), Excel Options, Add-Ins. Select the Data Analysis Tool Pack and add it in!

- Highlight the data from Question 1.
- Under Data select data analysis, descriptive statistics. Enter the data cell range and for input just put a cell where you want the top line to appear. Check the box for summary statistics and hit enter!

3. To draw box plot: calculate the statistical functions QUARTILE(q1), MIN, MEDIAN, MAX and QUARTILE(q3) in that order for each data set. Arrange the results on an Excel worksheet.

6.4: Binomial distribution and Normal Distribution

Discrete probability distributions

Binomial Trials

1. There are a fixed number of independent trials n.
2. Each trial has only two (hence binomial) outcomes, either "success" or "failure".
3. For the trials, the probability of success,p is always the same, and the probability of failure, $q = 1 - p$, is also always the same.
4. The expected value $E(X) = np$.

Excel Activity

Goal: Get a "feel" for binomial distributions by finding their probability distribution tables and graphing them.

Calculate the probability distribution table for X, a binomial distribution with 10 trials and probability of success p = 0.02. Use the drag feature to save yourself from a lot of typing!

X	P(X = x)
0 (say this is in cell A2)	=BINOMDIST (A2, 10,0.2,False)
1	
2	

Use Chart Wizard to plot the probabilities as a histogram (bar chart with no gaps!) You'll need to click on the bars of the chart and Select Data to get the 0, 1, 2, … as the X-axis labels and you'll need to select Format Data Series to remove gaps.

Repeat for n=10, p= 0.5 and n=10, p = 0.9. You'll get 3 tables and 3 histograms. What are the shapes of each distribution?

Answer the following: For small n Binomial Histograms tend to be ______ skewed if p < 0.5 and ______ skewed if p > 0.5.

Continuous probability distribution

6.E: Introduction to Statistics (Exercises)

CHAPTER OVERVIEW

7: Rational Reasoning

Learning Objectives

Develop the students:

- rational reasoning
- ability to work with Egyptian fractions.

Topic hierarchy

7.1: Dimensional Analysis
7.2: Egyptian Fractions
7.E: Rational Reasoning (Exercises)

Contributor

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7.1: Dimensional Analysis

Thinking out loud

Small pizza is 8" and large Pizza is 16". Are we getting the same amount of pizza if we order two small pizzas instead of one large one?

We all have faced a situation when need to be able to change from one unit of measurement to another unit of measurement. In this section, we discuss a method of converting units called dimensional analysis.

1 day = 24 hours,

1 hour = 60 minutes, and
1 minute = 60 seconds.

Imperial measurement

1 inch

1 foot= 12 inches

1 yard= 3 feet=36 inches

1 mile =5289 feet

Metric

millimeter (mm)

centimeter (cm), 1 cm = 10 mm

meter (m) , 1m= 100 cm= 1000 mm

kilometer (km). 1 km= 1000 m

Definition

A unit ratio is a fraction that has a value of 1 if both the numerator and the denominator are expressed in the same units.

Example 7.1.1:

If the speed limit says 90 kilometres per hour, what is the speed in miles per hour?

We can use dimensional analysis to convert this speed to miles per hour. We can also use reasoning to deduce that we need to divide 90 by 1.6.

Area

Example 7.1.1:

John and James have decided to pull up their old carpet and buy a new carpet. The room measures 15 feet by 11 feet, so the area is 165 square feet. However, when they go to the carpet store, they find that the prices are in square yards. How many square yards is their floor?

Maximizing area

Temperature

Temperature: In 1714, a German instrument maker named Gabriel Fahrenheit made the first mercury thermometer. He designated the lowest temperature he could create in the laboratory as 0° and the normal temperature of the body as 98°. On his scale, the freezing point of water is 32° and the boiling point of water is 212°.

Metric system

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7.2: Egyptian Fractions

This page is a draft and is under active development.

The Egyptians used a pictorial number system, with different symbols for every power of 10 to 1 000 000.

alt alt

1 = |

4 = ||||

10 = ∩

23 = ∩ ∩|||

100 =

112 = ∩||

Egyptian fractions can only have 1 as the numerator: $\frac{1}{4}, \frac{1}{16}, \frac{1}{137}$ etc. This can also be called unit fractions. (They also used $\frac{2}{3}$ but we will ignore this for now). The unit fraction is made by writing the number with a "mouth" symbol over the top –

$\frac{1}{2} =$

Egyptians did not like repeating fractions, therefore, each fraction must be unique. As a result, any fraction with numerator > 1 must be written as a combination of some set of Egyptian fractions. As a result of this mathematical quirk, Egyptian fractions are a great way to test student understanding of adding and combining fractions with different denominators (grade 5-6), and for understanding the relationship between fractions with different denominators (grade 5). We can use manipulative to explore Egyptian fractions. For a detailed activity, please see Activity.

Eye of Honus

alt

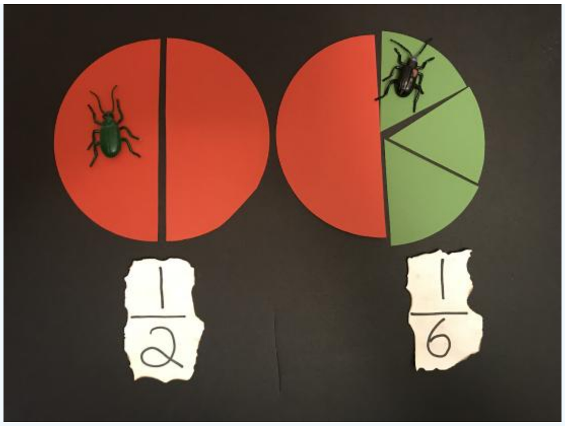
Party Time

Example 7.2.1: Party Time

You're having a pie party. Unfortunately, you forgot to cut the pies before the party started. How do you cut and split the pies evenly amongst your guests, if there are 3 people and 2 pies $\frac{2}{3}$? How do you split the pies using Egyptian fractions?

Solution

Using our manipulative, we begin by laying out 2 full pies. The biggest Egyptian fraction that can split these pies is $\frac{1}{2}$. This gives us $\frac{1}{2}$ of a pie for each person and $\frac{1}{2}$ of a pie leftover. We can split this last half piece into thirds again, giving us $\frac{1}{6}$ of a pie for each person. That is $\frac{2}{3} = \frac{1}{2} + \frac{1}{6}$.



Example 7.2.2:

If we were asked to write $\frac{3}{5}$ into a sum of Egyptian fractions, we could write $\frac{1}{5}+\frac{1}{5}+\frac{1}{5}$, since each fraction has a numerator of 1. The only problem with this is that they aren't unique fractions, therefore it wouldn't work for the sum of Egyptian fractions.

$\frac{3}{5}=\frac{1}{2}+\frac{1}{10}$.

Example 7.2.3:

What if there were only 4 pizzas to be split amongst 8 friends?

$\frac{5}{8}=\frac{1}{2}+\frac{1}{8}$.

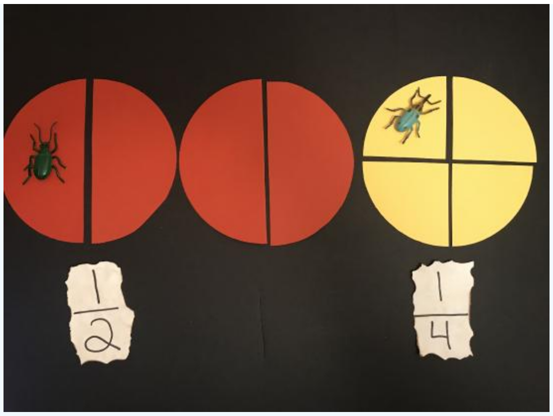
Exercise 7.2.1:

You're having a pie party. Unfortunately, you forgot to cut the pies before the party started. How do you cut and split the pies evenly amongst your guests?

1. There are 4 people and 3 pies $\frac{3}{4}$. How do you split the pies using Egyptian fractions?
2. There are 6 people and 5 pies $\frac{5}{6}$. How do you split the pies using Egyptian fractions?

Answer

1. Using our manipulative, we begin by laying out 3 full pies. The biggest Egyptian fraction that can split these pies is $\frac{1}{2}$. This gives us $\frac{1}{2}$ a pie for each person and a whole pie leftover. We can split this pie into 4 equal pieces, giving us $\frac{1}{4}$ of a pie for each person.



2. Using our manipulative, we begin by laying out 5 full pies. The biggest Egyptian fraction that can split these pies is $\frac{1}{2}$. This gives us $\frac{1}{2}$ a pie for each person and two whole pie leftovers. We can split these two pies into 6 equal pieces, giving us $\frac{1}{3}$ of a pie for each person.

Bigger Parties, Bigger Problems

You are now having an even bigger party with a lot more pies. Can you find the sum of Egyptian fractions that will help split the pies evenly amongst your guests? Having manipulative for these next questions would be unrealistic as there would be too many pies. Instead, we can think about the factors of 100 and use these as our denominators of our Egyptian fractions.The factors of 100 are $1, 2, 4, 5, 10, 20, 25, 50$, and 100.

Example 7.2.4:

There are 100 people and 91 pies $\frac{91}{100}$. How do you split the pies using Egyptian fractions?

Solution

Looking at our factors of 100, we can begin by subtracting the biggest Egyptian fraction $\frac{1}{2}$.

$$\frac{91}{100} - \frac{1}{2} = \frac{91}{100} - \frac{50}{100} = \frac{41}{100} \quad .$$

Next, we subtract $\frac{1}{4}$.

$$\frac{41}{100} - \frac{1}{4} = \frac{41}{100} - \frac{25}{100} = \frac{16}{100} \quad .$$

Notice that we can't subtract Next, we subtract $\frac{1}{5}=\frac{20}{100}$. Therefore, we subtract $\frac{1}{10}$.

$\frac{16}{100}-\frac{1}{10}=\frac{16}{100}-\frac{10}{100}=\frac{6}{100}$.

Next we subtract $\frac{1}{20}$.

$\frac{6}{100}-\frac{1}{20}=\frac{6}{100}-\frac{5}{100}=\frac{1}{100}$.

Thusm $\frac{91}{100}=\frac{1}{2}+\frac{1}{4}+\frac{1}{5}+\frac{1}{100}$. Each person would get $\frac{1}{2},\frac{1}{4},\frac{1}{5}$. and $\frac{1}{100}$ of a pie.

Exercise 7.2.2:

There are 100 people and 99 pies $\frac{99}{100}$. How do you split the pies using Egyptian fractions?

Answer

$\frac{99}{100}=\frac{1}{2}+\frac{1}{4}+\frac{1}{5}+\frac{1}{25}$. Also note that it can be spilit into Egyptian fractions which do not have their denominators as factors of 100. $\frac{99}{100}=\frac{1}{2}+\frac{1}{3}+\frac{1}{10}+\frac{1}{20}+\frac{1}{150}$.

Theorem 7.2.1

Any fraction $\frac{2}{n}$, where n is odd, can be expressed as a sum of two Egyptian fractions.

Proof

coming soon.

- Thanks to Jillian Periliat for all the diagrams.

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7.E: Rational Reasoning (Exercises)

Exercise $7.E.1$

Convert 6 feet 4 inches to centimetres. Fact: 1 inch $=2.54$ cm

Answer

$76 * 2.54\text{cm}$

Exercise $7.E.2$

Convert 850 square feet to square meters. Why do you think that Real-estate sites like to list house area regarding square feet instead of square meters? Fact: 1 foot = 0.3048m

Exercise $7.E.3$

John claims that the surface area of a cone is given by the formula: $A = \pi r\sqrt{r+h}$

where r is the radius of the cone and h is the height of the cone. How can you convince John, that he must be wrong without resorting to showing her formula in a textbook?

Exercise $7.E.4$

Yahoo Autos cites that the fuel efficiency of the 2008 Toyota Prius is 4L/100 km in the City and 4.2 L/100 km on the highway. American site states that Toyota's 2008 Prius hybrid car uses an average of 48 miles per gallon in city driving, and 45 mpg on the highway. Do these figures agree? According to the Canadian figures, if I spend about $20 per week in a city driving in a 2008 Prius, roughly how many kilometres have I travelled?

Exercise $7.E.5$

At what temperature do Celsius and Fahrenheit agree?

Exercise $7.E.6$

The exterior dimensions of a freezer are 48 inches by 36 inches by 24 inches, and it is advertised as being 27.0 cubic ft. Is the advertised volume correct?

Exercise $7.E.7$

Which holds more soup: a can with a diameter of 3 inches and a height of 4 inches or a can with a diameter of 4 inches and a height of 3 inches?

Exercise $7.E.8$

A larger cube has a volume of 81 m^3. A smaller cube has the length of the edges one-third of the length of the edges of the larger cube. What is the volume of the smaller cube?

Exercise $7.E.9$

A larger equilateral triangle was created using four smaller equilateral triangles as shown in the figure. The perimeter of the smaller triangle is 18 cm, then what is the perimeter of the larger equilateral triangle?

Exercise $7.E.10$

There are 100 people and 97 pies $\frac{97}{100}$. How do you split the pies using Egyptian fractions?

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